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RESPONSE OF NONLINEAR SYSTEMS TO UNUSUAL INPUTS

José António Joglar

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THESIS

RESPONSE OF NONLINEAR SYSTEMS
TO UNUSUAL INPUTS

by

José António Joglar

December 1974

Thesis Advisor: O.M. Baycura

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The aim of this work is to test this technique in obtaining the solution of several time varying linear equations and to produce a computer program that improves the solution accuracy and ease of use.

Response of Nonlinear Systems to Unusual Inputs

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Submitted in partial fulfillment of the
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Handling nonlinear systems is not a simple task, especially if a computer is not available. Generally, although a precise result for every case would be preferable, an approximate solution is usually sufficient for most engineering solutions.

R. Boxer and S. Thaler developed a technique [1] presented in Proceedings of IRE that allow to obtain the response of linear and nonlinear systems, without knowledge of the roots of the system characteristic equation.

The aim of this work is to test this technique in obtaining the solution of several time varying linear equations and to produce a computer program that improves the solution accuracy and ease of use.

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I. INTRODUCTION

Two basic methods exist for solving the response of a system which generally is presented as an integro-differential equation: analytic methods and numerical methods.

For simple problems, analytic methods give precise solutions and have the advantage that these solutions can be given as a function of the parameters of the system.

As the order of the differential equation increases, numerical methods give less labor in providing a solution and can be, at some extent the only method available.

Laplace transform is limited to the case of linear differential equations. Nonlinear differential equations are out of the scope of this technique, preventing its application.

Even sometimes with linear systems, if the inverse integral is complex, it is difficult and sometimes impossible to find analytically the time function.

However, it is possible to approximate the Laplace transform of a linear system by a z-transform, obtaining thereafter a time series representing the values of the response at equally spaced intervals of time, by synthetic division.

The method [1] described and used in this paper utilizes the z-transform directly in order to obtain the solution of the linear and nonlinear systems.

In this paper, an explanation is given of this method applied to linear systems and its extension to time-varying and nonlinear systems.

A computer was used for faster and more reliable results, but the method described is compatible with handwork.

C.P.U. time in the computer, for this technique, is much smaller than other numerical methods due to the less number of iteration steps needed. This technique could reduce solution times by several orders of magnitude over conventional numerical methods.

II. PRESENTATION OF THE METHOD

A. PROCEDURE

The basic procedures of the method are given, before the description or the theory, to show its simplicity.

Suppose that one has the Laplace transform of an output response. It will be usually in the following form:

$$F(s) = \frac{a_o + a_1 s + a_2 s^2 + \cdots + a_k s^k + \cdots + a_n s^n}{b_o + b_1 s + b_2 s^2 + \cdots + b_k s^k + \cdots + b_m s^m} \qquad (1)$$

The steps to obtain the time response are:

1. Express the function F(s) as a rational fraction in powers of s^{-1} by dividing the numerator and denominator by s^m.
2. Substitute for s^{-k} a rational fraction in powers of z^{-1} obtained from Table I and rearrange F(s) as a rational fraction in powers of z^{-1}.
3. Divide the resulting expression by T, where T is the time interval between points at which the result is desired.
4. Expand the fraction by synthetic division into a series of the form:

$$D_o + D_1 z^{-1} + D_2 z^{-2} + \cdots + D_n z^{-n} + \cdots \qquad (2)$$

where D_n, the coefficient of z^{-1}, is the approximate value of the time response at $t = nT$.

B. LINEAR SYSTEMS

Take an arbitrary function f(t) as in Fig. 1(a). Its Laplace transform and inverse Laplace transform are defined as:

$$F(s) = \int_0^\infty f(t)\, e^{-st}\, dt \tag{3}$$

$$f(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} f(s)\, e^{st}\, ds \tag{4}$$

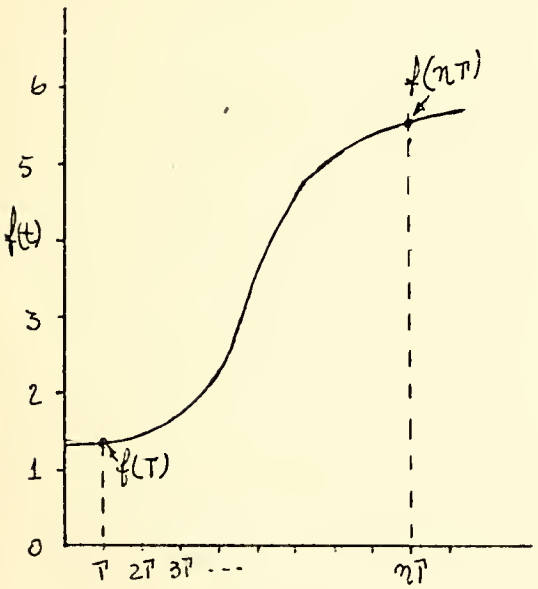


Fig. 1(a): Arbitrary f(t)

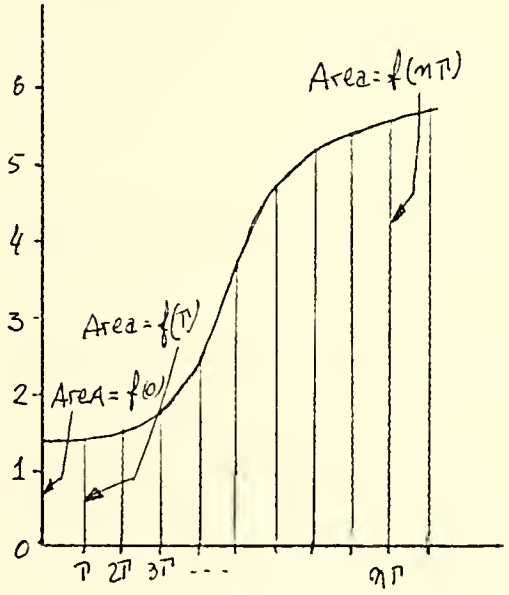


Fig. 1(b): Sampled f(t)

Figure 1(b) shows f(t) sampled by shifted unit impulses. The result is a series of impulses at t = nT.

Applying Laplace to the sampled function one gets:

$$F^*(e^{st}) = \sum_{n=0}^{\infty} f(nt)\, e^{-snT} \tag{5}$$

substituting in (5) e^{sT} by z so:

$$z = e^{sT} \tag{6}$$

the result obtained is known as the z-transform:

$$F^*(z) = \sum_{n=0}^{\infty} f(nT)\, z^{-n} \tag{7}$$

and the inverse is given by:

$$f(nT) = \oint F^*(z)\, z^{n-1}\, dz \tag{8}$$

where the contour is a circle centered at the origin and including all the singularities of $F^*(z)$.

The equation (6) maps the $j\omega$ axis of the s-plane into a unit circle around the origin in the z-plane.

There are other methods for obtaining direct and inverse z-transforms.

In closed form the function will appear as:

$$F^*(z) = \frac{a_o + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_n z^{-n}}{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_m z^{-m}} \tag{9}$$

Division of the numerator by the denominator, will result in the coefficients of f(nT) as defined in (7), without the use of the inversion integral formula (8).

In order to apply this useful characteristic of z-transform theory to non-sampled systems one uses the inverse Laplace integral (4),

$$f(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(s)\, e^{st}\, ds$$

which can be expressed piecewise as:

$$f(t) = \frac{1}{2\pi j} \int_{-J\pi/T}^{J\pi/T} f(s)e^{st}ds + \frac{1}{2\pi j} \int_{J\pi/T}^{J\infty} [F(s)e^{st} + f(-s)e^{-st}]ds \qquad (10)$$

The way the contour is divided is shown in Fig. 2. The first integral gives the part of the contour shown between $-(\pi/T) \leq \omega \leq \pi/T$. The remaining contour $\omega > (\pi/T), \omega < -(\pi/T)$ is given by the second integral.

If 1/T is sufficiently large, a good approximation will be held by the first integral of (10) and the second integral may be discarded, being the resulting error given by:

$$e(t) = \frac{1}{2\pi j} \int_{-J\pi/T}^{J\infty} f(s)e^{st} + F(-s)e^{-st}\, ds \qquad (11)$$

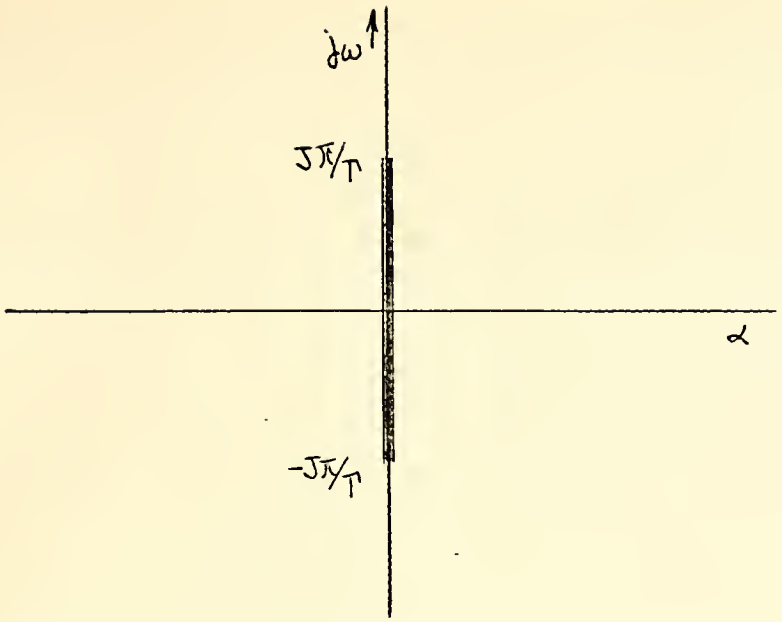


Figure 2: Contour in s-plane for approximate solution

The approximation, $f_A(t)$, of $f(t)$ is given by:

$$f_A(t) = \frac{1}{2\pi j} \int_{-J\pi/T}^{J\pi/T} F(s)\, e^{st}\, ds \qquad (12)$$

Making $t = nT$

$$f_A(nt) = \frac{1}{2\pi j} \int_{-J\pi/T}^{J\pi/T} F(s)\, e^{snT}\, ds \qquad (13)$$

Solving eq. (6) for s, one obtains:

$$s = 1/T \ln z \qquad (14)$$

Making the substitution of (14) into (13) the following equation is obtained

$$f_A(nT) = \frac{1}{2\pi j} \int_{-J\pi/T}^{J\pi/T} F(1/T \ln z)\, e^{nT(1/T)\ln z}\, d(1/T \ln z) \tag{15}$$

$$f_A(nT) = \frac{1}{2\pi j} \oint \frac{1}{T} F(1/T \ln z)\, z^{n-1}\, dz \tag{16}$$

where the contour, again, is the unit circle in the z-plane.

If eq. (16) is compared with eq. (8), it is seen that it has the form of an inverse z-transform except for the constant multiplier 1/T.

On the other hand, as F(1/T ln z) is a transcendental function, it is not expandable directly into a series of powers of z^{-1} by synthetic division. The synthetic division technique can, however, be used if a proper approximation could be found.

If F(s) is expanded in a series of descending powers of s like

$$F(s) = a_1 s^{-1} + a_2 s^{-2} + \cdots \tag{17}$$

a solution of the form

$$f(t) = a_1 + a_2 t + a_3 \frac{t^2}{2!} + \cdots \tag{18}$$

can be obtained, but this solution is not always valid.

The way to follow is to rearrange F (1/T)/ln z as a rational function of two polynomials in powers of $s^{-1} = T/\ln z$.

For this purpose, the power series for ln z used is:

$$\ln z = 2[u + 1/3 \cdot u^3 + 1/5 \cdot u^5 + \cdots] \tag{19}$$

where:

$$u = \frac{1 - z^{-1}}{1 + z^{-1}}$$

So the following equation is obtained:

$$\frac{1}{s} = \frac{T}{\ln z} = \frac{T/2}{u + 1/3\ u^3 + 1/5\ u^5} \tag{20}$$

By synthetic division the following Laurent series is obtained:

$$\frac{1}{s} = \frac{T}{\ln z} = \frac{T}{2}\left[\frac{1}{u} - \frac{1}{3}u - \frac{4}{45}u^3 - \frac{44}{945}u^5 + \cdots\right] \tag{21}$$

To obtain series expansion of s^{-k}, bothsides of (21) can be raised to the k^{th} power.

Table I (on page 15) was obtained by using only the principal part and the constant term of the Laurent series. The result is of the form

TABLE I

s^{-k}	z-Form = $F_k(z^{-1})$
s^{-1}	$\frac{T}{2}\frac{1+z^{-1}}{1-z^{-1}}$
s^{-2}	$\frac{T^2}{12}\frac{1+10z^{-1}+z^{-2}}{(1-z^{-1})^2}$
s^{-3}	$\frac{T^3}{2}\frac{z^{-1}+z^{-2}}{(1-z^{-1})^3}$
s^{-4}	$\frac{T^4}{6}\frac{z^{-1}+4z^{-2}+z^{-3}}{(1-z^{-1})^4}-\frac{T^4}{720}$
s^{-5}	$\frac{T^5}{24}\frac{z^{-1}+11z^{-2}+11z^{-3}+z^{-4}}{(1-z^{-1})^5}$

$$s^{-k} \quad \frac{Nk\ (z^{-1})}{(1 - z^{-1})^k} = F_k(z^{-1}) \tag{22}$$

where N_k is a polynomial in powers of z^{-1}. The right hand side of (24) is called the "z-form" for s^{-k}.

Example 1 – Third Order System (Fig. 3)

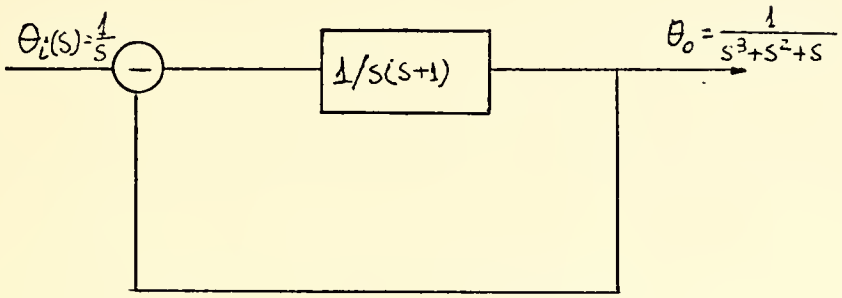


Fig. 3: Typical Feedback System for Example 1

With a step input, as shown, the Laplace transform of the output is:

$$\theta_0(s) = \frac{1}{s^3 + s^2 + s} \tag{23}$$

Following the step by step procedure as given in the Introduction,

$$\theta_0(s) = \frac{s^{-3}}{1 + s^{-1} + s^{-3}} \tag{24}$$

Now, substituting the correspondent forms of s^{-k} from Table I and dividing the result by T:

$$\theta_{OA}^*(z) = \frac{6T^2(z^{-1} + z^{-2})}{(12+6T+T^2) - (36+6T-9T^2)z^{-1} + (36-6T-9T^2)z^{-2} - (12-6T+T^2)z^{-3}} \tag{25}$$

The solution is obtained by choosing T, the time interval at which rate our solution is obtained, and proceeding in dividing the denominator into the numerator.

Let us assume T = 1.0, resulting in the following output:

$$\theta_{OA}^*(z) = \frac{6z^{-1} + 6z^{-2}}{19 - 33z^{-1} + 21z^{-2} - 7z^{-3}} \tag{26}$$

By long division, the result is:

$$.316z^{-1} + .864z^{-2} + 1.15z^{-3} + 1.16z^{-4} + 1.06z^{-5} + .987z^{-6} + \cdots$$

The accuracy improves if smaller value for T is chosen. Letting T = 0.5 in this same example, substituting on eq. (25):

$$\theta_{OA}^*(z) = \frac{1.5z^{-1} + 1.5z^{-2}}{15.25 - 36.75z^{-1} + 30.75z^{-2} - 9.25z^{-3}} \tag{27}$$

carrying out the long division, the result is:

$$0.984z^{-1} + .335z^{-2} + .610z^{-3} + .853z^{-4} + \cdots$$

The points obtained are plotted together with the precise solution in Fig. 4.

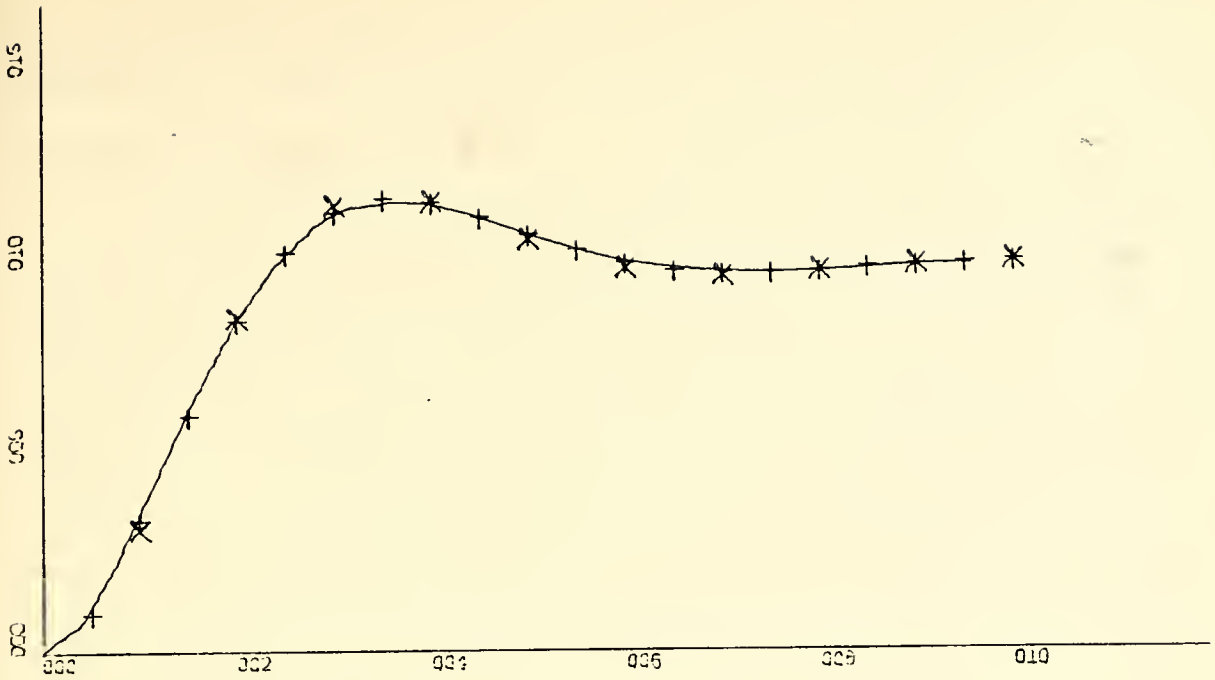


Figure 4

X-scale = 2.0 units/inch
Y-scale = .5 units/inch

— Precise solution
x Approximate solution for $\Delta t = 1.0$
\+ Approximate solution for $\Delta t = 0.5$

Example 2 — Solution of Linear Differential Equation

To use the present technique in obtaining the solution of integro-differential equations, one takes the Laplace transform of the equation, initial conditions included as usual, and then proceeds as before. However, after transforming the equation, the following rules can be applied:

1. If the degree of the denominator of the Laplace transform is higher than the degree of the numerator by two or more, proceed directly.

2. If the degree of the numerator is one less, equal to, or greater than the degree of the denominator, divide until the remainder meets the criterion of Rule 1.

Using the procedure stated in the rules above one eliminates the high frequency from F(s), that is implied if the numerator is one degree less than the denominator.

To illustrate this, a simple example is shown:

$$\frac{dY}{dt} + Y = 1 \quad . \tag{28}$$

Assuming for initial condition Y(0) = u, the Laplace transform Y(s) is:

$$Y(s) = \frac{1 + s \cdot u}{s(s + 1)} = \frac{s \cdot u + 1}{s^2 + s} \tag{29}$$

Using Rule 2 above, one gets:

$$Y(s) = \frac{u}{s} + \frac{1 - u}{s^2 + s} \tag{30}$$

In this procedure, the step function of amplitude u is added to the result obtained after operating upon $(1 - u)/(s^2 + s)$.

Now, letting u = 0.1, and expressing the second term of eq. (30) in powers of s^{-1},

$$Y_1(s) = \frac{0.9s^{-2}}{1 + s^{-1}} \tag{31}$$

Substituting the z-forms of Table I (on page 15) and dividing by T,

$$Y_{1A}^*(z) = \frac{0.9T(1 + 10z^{-1} + z^{-2})}{(12 + 6T) - 24z^{-1} + (12 - 6T)z^{-2}} \tag{32}$$

Letting T = 0.1 and carrying out the long division one gets:

$$.00714 + .085z^{-1} + .163z^{-2} + .233z^{-3} + .296z^{-4} + \cdots \tag{33}$$

This result is plotted in Fig. 5 after adding the step function of amplitude 0.1.

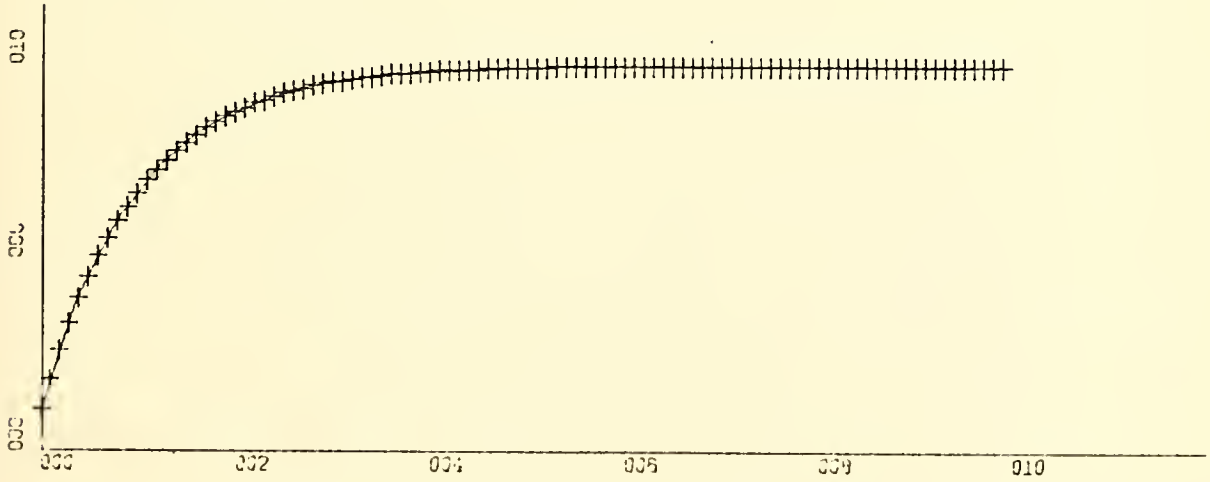


Figure 5

X-scale = 2.0 units/inch
Y-scale = .5 units/inch

— Precise solution
\+ Approximate solution

C. TIME-VARYING SYSTEMS

A linear differential equation with time-varying coefficients may be expressed as

$$\frac{d^n}{dt^n}\left[f_n(t)Y\right] + \frac{d^{n-1}}{dt^{n-1}}\left[f_{n-1}(t)Y\right] + \cdots + f_0(t)Y = f(t) \qquad (34)$$

where $f_n(t)$, $f_{n-1}(t),\ldots,f_0(t)$ are functions of the independent variable t.

To solve this type of equation the following steps may be executed:

1. Obtain the Laplace transform of (34) in the usual manner including initial conditions.

2. Proceed as in the constant coefficient case discussed above. In performing the long division process, however, change the values of $C_n,\ldots,C_0$ at each step in the division process to the values at the corresponding times of the functions

$$\begin{aligned} C_n &= f_n(t) \\ &\vdots \\ C_0 &= f_0(t) \end{aligned} \qquad (35)$$

In general

$$C_i = f_i(t) \qquad (36)$$

D. NONLINEAR SYSTEMS

A similar manner like used in II.C. may be applied to nonlinear differential equations. The regression equation is obtained by long division process as in the time varying example.

However, the factors C_n, C_{n-1}, ..., C_0 must now represent functions of the dependent variable.

Generally, a nonlinear differential equation can be represented as:

$$\frac{d^n}{dt^n}\left[f_n(Y,\frac{dY}{dt},\ldots)Y\right] + \frac{d^{n-1}}{dt^n}\left[f_{n-1}(Y,\frac{dY}{dt},\ldots)Y\right] + \cdots$$

$$+ \cdots + f_o(Y,\frac{dY}{dt},\ldots)Y = f(t) \qquad (37)$$

being the rules for the procedure as follow:

1. Obtain the Laplace transform of (37) in the usual manner including initial conditions.

2. Proceed as in the constant coefficient case. In performing the long division process, however, change the value of $C_n \cdots C_o$ at each step in the division process to the most recent values available for

$$\begin{aligned} C_n &= f_n(Y,\frac{dY}{dt},\ldots) \\ C_{n-1} &= f_{n-1}(Y,\frac{dY}{dt},\ldots) \\ &\vdots \\ C_o &= f_o(Y,\frac{dY}{dt},\ldots) \end{aligned} \qquad (38)$$

In general

$$C_i = f_i(Y, \frac{dY}{dt}, \ldots) \tag{39}$$

In other words, in the first step, the values of C_i are obtained from initial conditions for Y, $\frac{dY}{dt}$, ... The result of the first division is a new value of Y to be used in calculating C_n, C_{n-1}, ..., C_o for the second step in the division. New values to be used for the derivatives of Y with respect to t are calculated from the new values of the dependent variables. The process is difficult to describe in words, but is simple in concept.

To illustrate the procedure an example is given:

Example 3:

Let $$\frac{dY}{dt} + Y^2 = 1 \tag{40}$$

Carrying out the procedure outlined above, eq.(40) is written

$$\frac{dY}{dt} + C \cdot Y = 1 \tag{41}$$

The Laplace transform, assuming C to be constant and expressing in powers of s^{-1} is

$$Y(s) = \frac{s^{-2}}{1 + Cs^{-1}} \tag{42}$$

Substituting the z-forms of Table I and dividing by T yields:

$$Y_A^*(z) = \frac{T(1 + 10z^{-1} + z^{-2})}{(12 + 6CT) - 24z^{-1} + (12 - 6CT)z^{-2}} \qquad (43)$$

Letting T = 0.1,

$$Y_A^*(z) = \frac{0.1 + 1.0z^{-1} + z^{-2}}{(12 + 0.6C) - 24z^{-1} + (12 - 0.6C)z^{-2}}$$

Carrying out the long division procedure yields:

$$.00833 + 0.100z^{-1} + 0.199z^{-2} + 0.295z^{-3} + 0.390z^{-4} + \ldots \qquad (44)$$

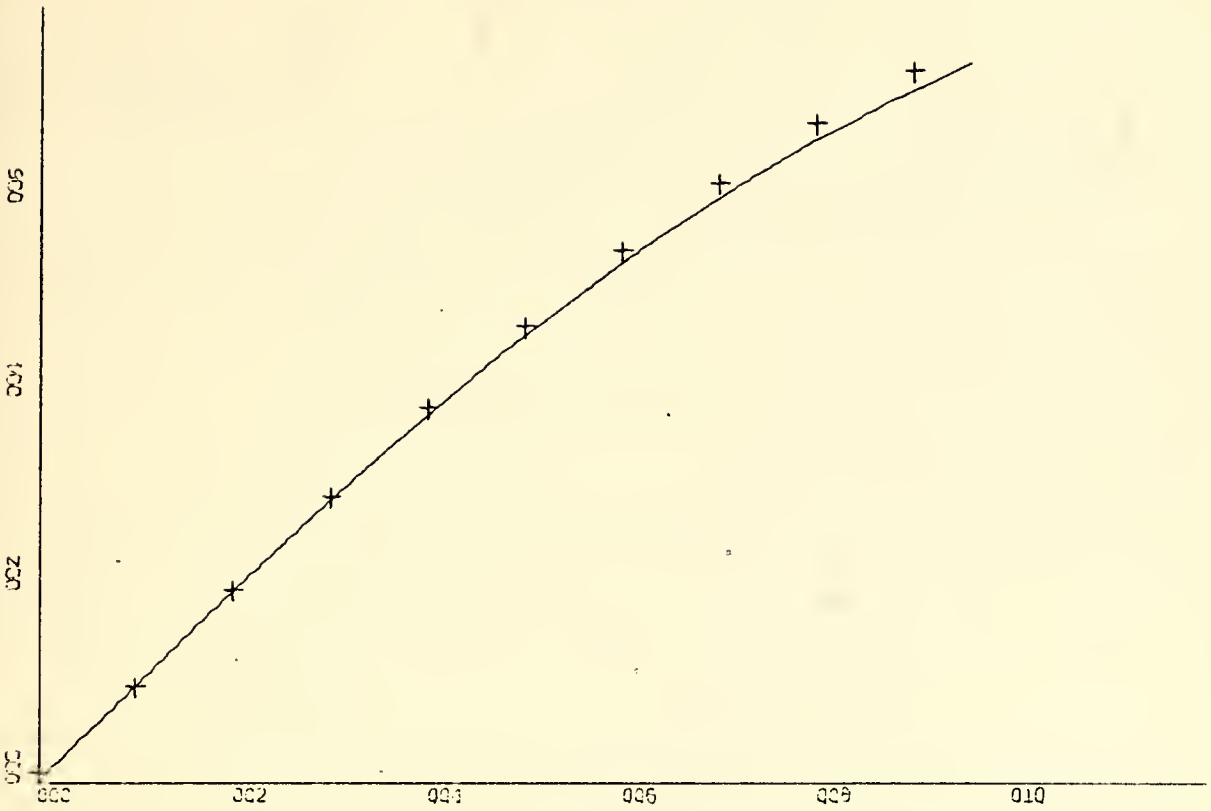


Figure 6

X-scale = 2.0 units/inch
Y-scale = .5 units/inch

— Precise solution
\+ Approximate solution

III. CASE STUDIES

A. INTRODUCTION

In order to test the present technique, a collection of linear differential equations with varying coefficients were studied, whose basic format is:

$$\dot{Y} + P(t)Y = Q(t)$$

where P(t) and Q(t) are time functions. For the purpose of this work, nine equations in all were processed successfully for what P(t) took the values $\sqrt{t}$, t and t^2, and Q(t) took the values $1 \cdot u(t)$, t and t^2.

The collection of equations, the case nubmers, and the sections in this work, are as follows:

$\dot{Y} + \sqrt{t}\,Y = 1$	Case 1	
$\dot{Y} + tY = 1$	Case 2	III B
$\dot{Y} + t^2Y = 1$	Case 3	
$\dot{Y} + \sqrt{t}\,Y = t$	Case 4	
$\dot{Y} + tY = t$	Case 5	III C
$\dot{Y} + t^2Y = t$	Case 6	
$\dot{Y} + \sqrt{t}\,Y = t^2$	Case 7	
$\dot{Y} + tY = t^2$	Case 8	III C
$\dot{Y} + t^2Y = t^2$	Case 9	

In Case 10, an example from Ref. 2 was studied, Comments on this problem are given in Case 10 discussion.

In cases 1 to 9 the same type of procedure was followed, applying directly the steps as in II A of this work and experimenting for time intervals form 0.2 sec. to 2.0 sec. in order to show the advantages of this technique with relatively large Δt's, its stability together with an algorithm for computer solution and application to hand calculation, i.e., small number of division steps for good results.

B. CASE 1 — $\dot{Y} + \sqrt{t}\,Y = 1$

The equation $\dot{Y} + P(t)Y = Q(t)$ is rewritten for this case as:

$$\frac{dY}{dt} + \sqrt{t}\,Y = 1 \qquad Y_{(t=0)} = 0$$

following the procedure, as in II A:

$$\frac{dY}{dt} + PY = 1 \qquad P = \sqrt{t}$$

the Laplace transform is

$$sy + Py = 1/s$$

rearranging and expressing in powers of s^{-1}:

$$s^2y + Psy = 1 \qquad y(s^2+Ps) = 1$$

$$y = \frac{1}{s^2 + Ps} = \frac{s^{-2}}{1 + Ps^{-1}}$$

Substituting the z-form values for s^{-k} from Table I (page 15)

$$y_A^*(z) = \frac{\frac{T^2}{12}\,\frac{1 + 10z^{-1} + z^{-2}}{(1 - z^{-1})^2}}{1 + P\,\frac{T}{2}\,\frac{1 + z^{-1}}{1 - z^{-1}}}$$

Dividing by T and rearranging:

$$y_A^*(z) = \frac{T + 10Tz^{-1} + Tz^{-2}}{(12 + 6PT) - 24z^{-1} + (12 - 6PT)z^{-2}} \qquad (45)$$

where again P is the time varying coefficient of Y in the differential equation and T is the iteration time, or in other words the time difference between results.

The following step consists of dividing the denominator into the numerator and substituting in P, its value for the corresponding time (t = nT). As the exponent of z for the first value in the quotient is zero, $P(t) = \sqrt{t}$ at t = 0, P(t) = 0 in the first division step.

The value for T first chosen was 0.2 sec. .

The algorithm used for computing the quotient (Program 1, page 30) as the solution, for values of T from 0.2 sec. up to 2 sec. with the "precise" solution in continuous line and with the approximate solution as crosses follow; the "precise" solution was obtained by computer, using the subroutine called INTEG1.

Scales in the graphs (Figs. 7 to 20) are the same in each case.

In the numerical output tables, the time is referred to as T, the approximate solution is referred to as Q(N) and the "precise" solution as PR. TFIN is the final time.

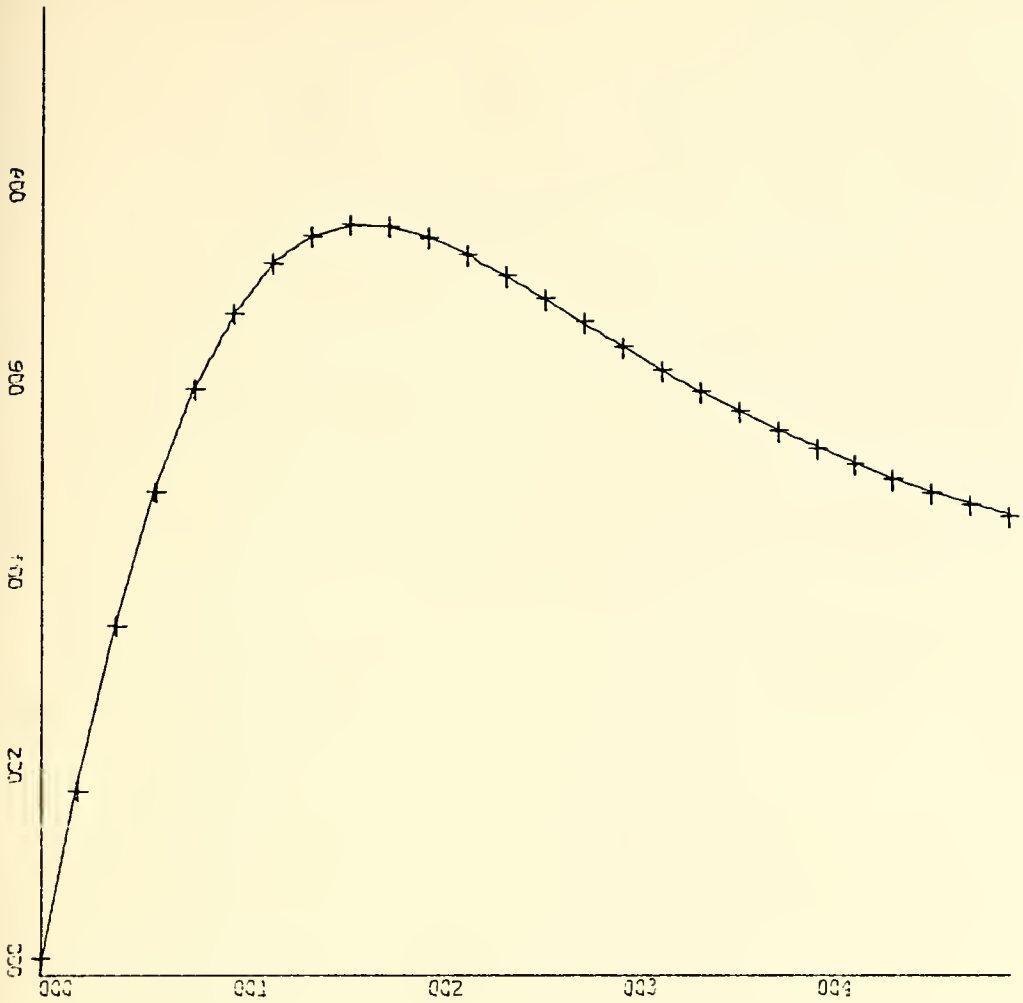
It can be seen that the approximated solution even for large time iterations is very close to the precise solution.

FOR DELTA T = 0.20 TFIN = 5.2

T	Q(N)	PR
0.0	0.01667	0.0
0.2	0.19144	0.19300
0.4	0.36010	0.36197
0.6	0.49870	0.50039
0.8	0.60588	0.60716
1.0	0.68335	0.68414
1.2	0.73455	0.73486
1.4	0.76372	0.76363
1.6	0.77528	0.77491
1.8	0.77345	0.77289
2.0	0.76192	0.76128
2.2	0.74384	0.74318
2.4	0.72170	0.72109
2.6	0.69743	0.69689
2.8	0.67245	0.67200
3.0	0.64773	0.64738
3.2	0.62392	0.62366
3.4	0.60141	0.60122
3.6	0.58039	0.58027
3.8	0.56092	0.56085
4.0	0.54297	0.54294
4.2	0.52648	0.52647
4.4	0.51132	0.51133
4.6	0.49738	0.49741
4.8	0.48454	0.48458
5.0	0.47268	0.47272

```
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
C   Q(T)=STEP , P(T)=SQRT(T)
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      INTEGER *4ITB(12)/12*0/
      REAL *4RTB(28)/28*0.0/
      DIMENSION A(100), Q(100), X(100), XX(100), PR(100)
      ITB(3) = 5
      ITB(4) = 5
C
      DO 1 I=1,26
      READ (5,5) XX(I),PR(I)
    1 CONTINUE
C
    2 READ (5,5,END=4) TD,TFIN
      WRITE (6,6) TD,TFIN
      WRITE (6,7)
      M = TFIN/TD
      A(1) = TD
      A(2) = 10.0*TD
      A(3) = TD
      A(4) = 0.0
C
      DO 3 N=1,M
      T = (N-1)*TD
C
      P = SQRT(T)
C
      F1 = 12.0+6.0*(P*TD)
      F2 = -24.0
      F3 = 12.0-6.0*(P*TD)
C
C     Q(N) IS THE OUTPUT
C
      Q(N) = A(N)/F1
      A(N+1) = A(N+1)-(Q(N)*F2)
      A(N+2) = A(N+2)-(Q(N)*F3)
      A(N+3) = 0.0
      X(N) = T
      WRITE (6,5) T,Q(N)
    3 CONTINUE
C
      WRITE (6,8)
      WRITE (6,5) (XX(I),PR(I),I=1,26)
      ITB(1) = 1
      ITB(2) = 0
      ITB(12) = 1
      CALL DRAWP (26,XX,PR,ITB,RTB)
      ITB(1) = 3
      ITB(2) = 2
      CALL DRAWP (M,X,Q,ITB,RTB)
      GO TO 2
    4 STOP
C
    5 FORMAT (2F10.5)
    6 FORMAT ('1','FOR DELTA T=',F6.2,4X,' TFIN=',F4.1)
    7 FORMAT ('  T=',7X,' Q(N)=')
    8 FORMAT ('  XX=',7X,'   PR=')
      END
```

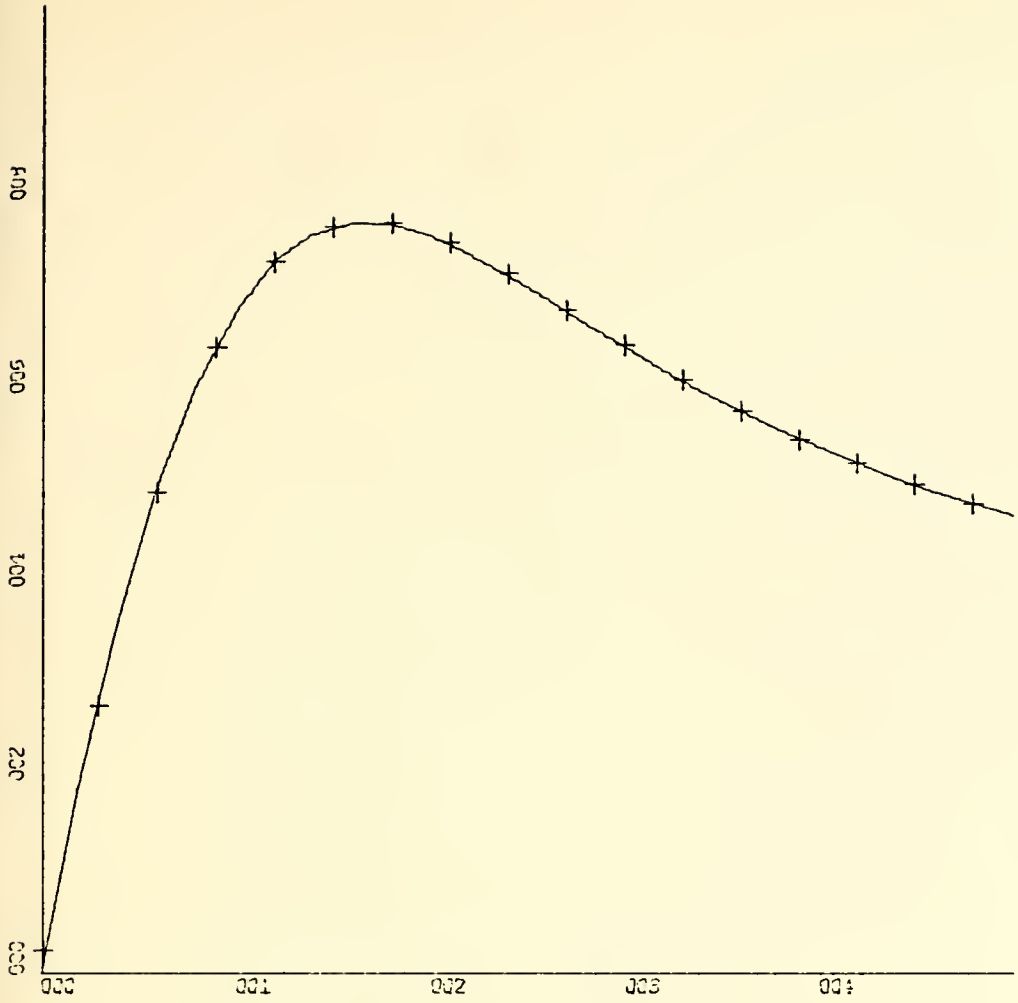
PROGRAM 1



x-scale = 1.0 units/inch
y-scale = 0.2 units/inch

FOR DELTA T = 0.2

FIGURE 7

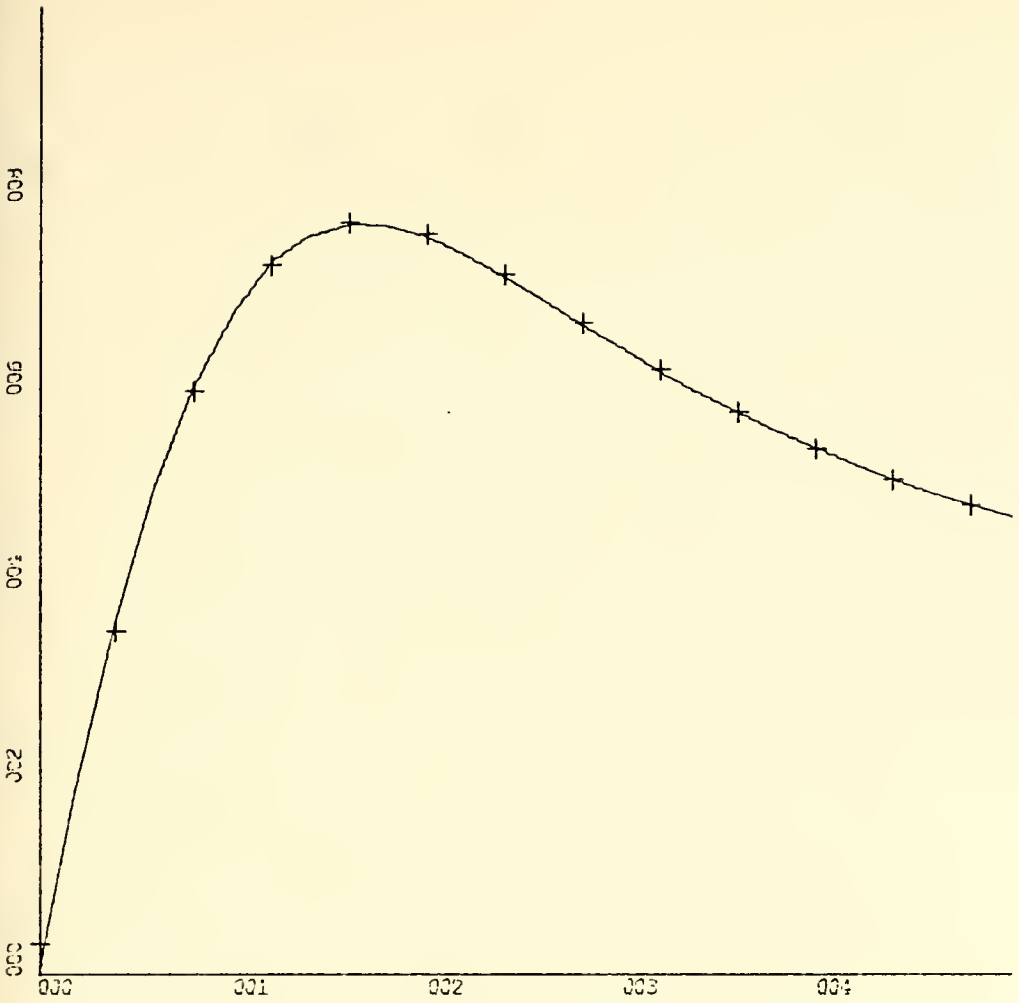


x-scale = 1.0 units/inch

FOR DELTA T = 0.3 TFIN = 5.2

T	Q(N)
0.0	0.02500
0.3	0.27722
0.6	0.49673
0.9	0.64695
1.2	0.73424
1.5	0.77180
1.8	0.77421
2.1	0.75441
2.4	0.72252
2.7	0.68563
3.0	0.64822
3.3	0.61282
3.6	0.58059
3.9	0.55187
4.2	0.52653
4.5	0.50422
4.8	0.48454

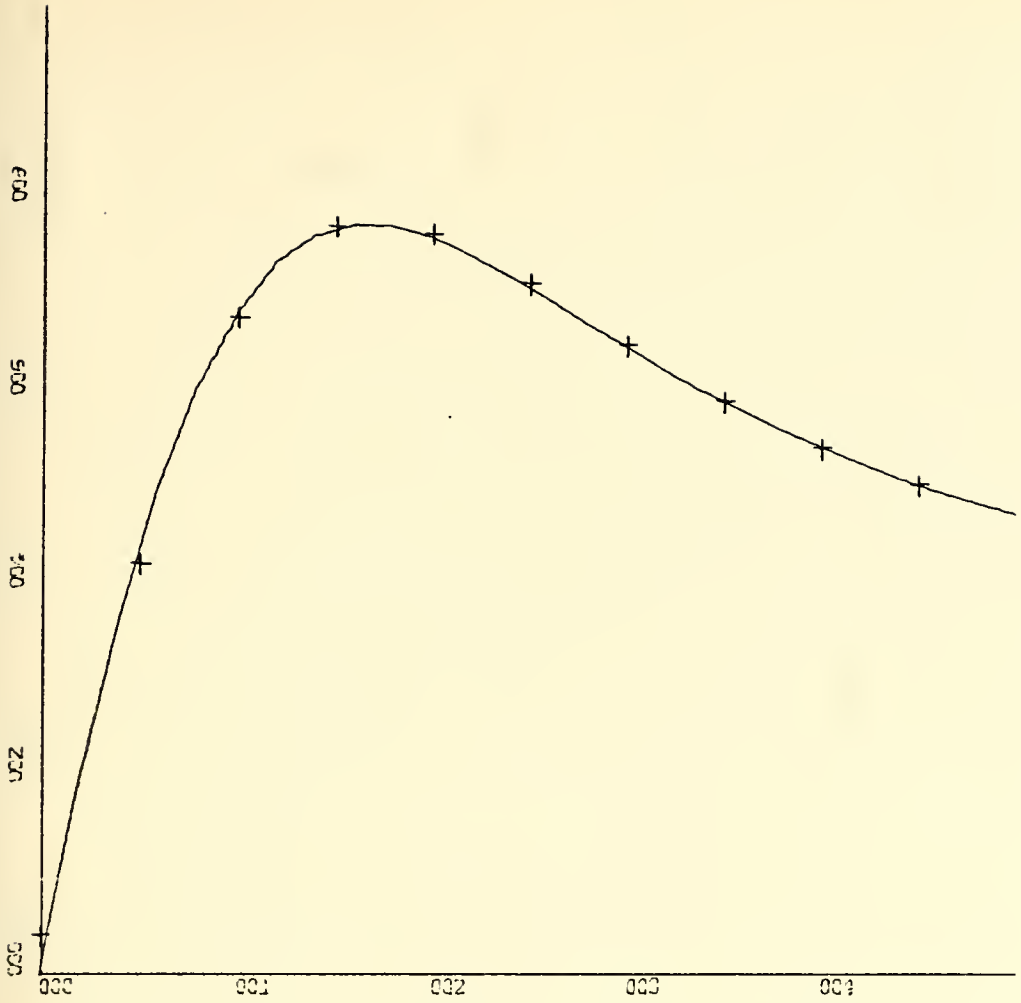
FIGURE 8



FOR DELTA T = 0.40 TFIN = 5.2

T	Q(N)
0.0	0.03333
0.4	0.35508
0.8	0.60241
1.2	0.73386
1.6	0.77661
2.0	0.76404
2.4	0.72370
2.8	0.67393
3.2	0.62484
3.6	0.58087
4.0	0.54317
4.4	0.51137
4.8	0.48453

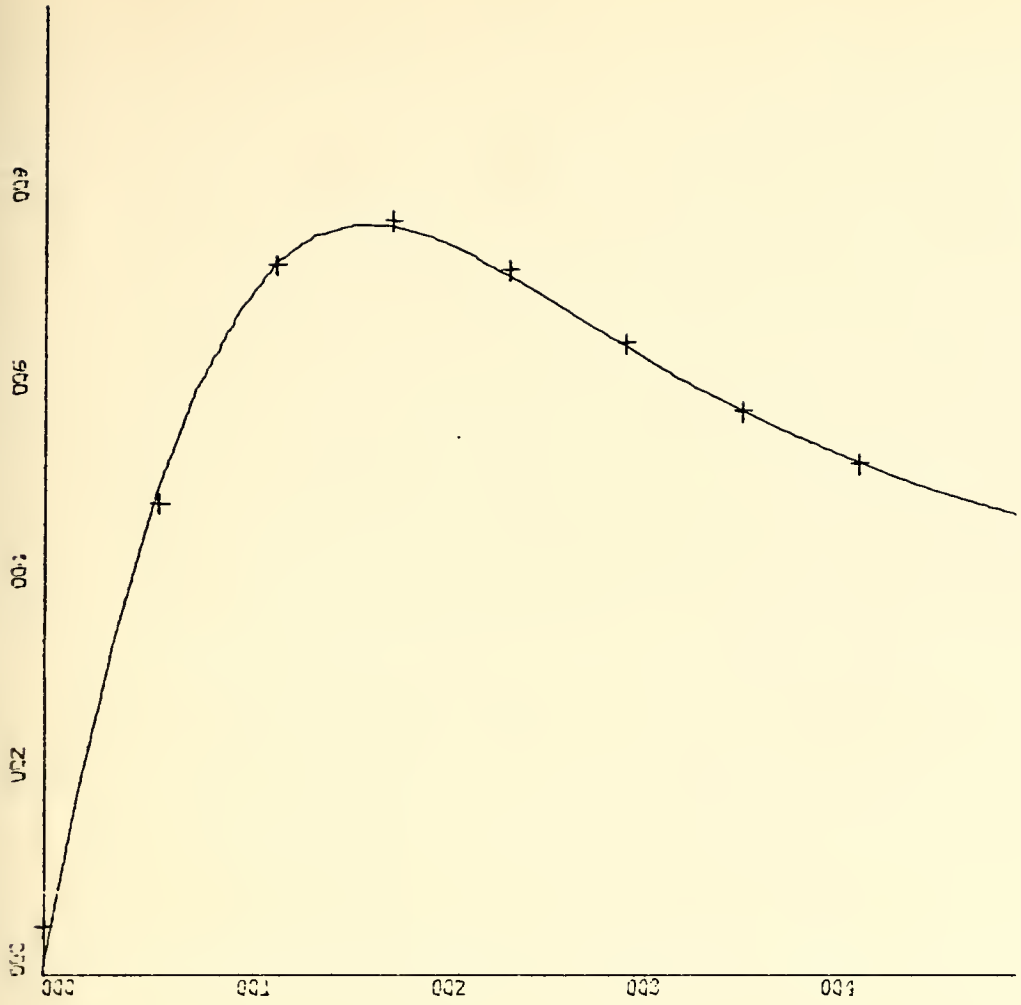
FIGURE 9



FOR DELTA T = 0.5 TFIN = 5.2

T	Q(N)
0.0	0.4167
0.5	0.42489
1.0	0.67982
1.5	0.77314
2.0	0.76570
2.5	0.71310
3.0	0.64984
3.5	0.59170
4.0	0.54331
4.5	0.50424

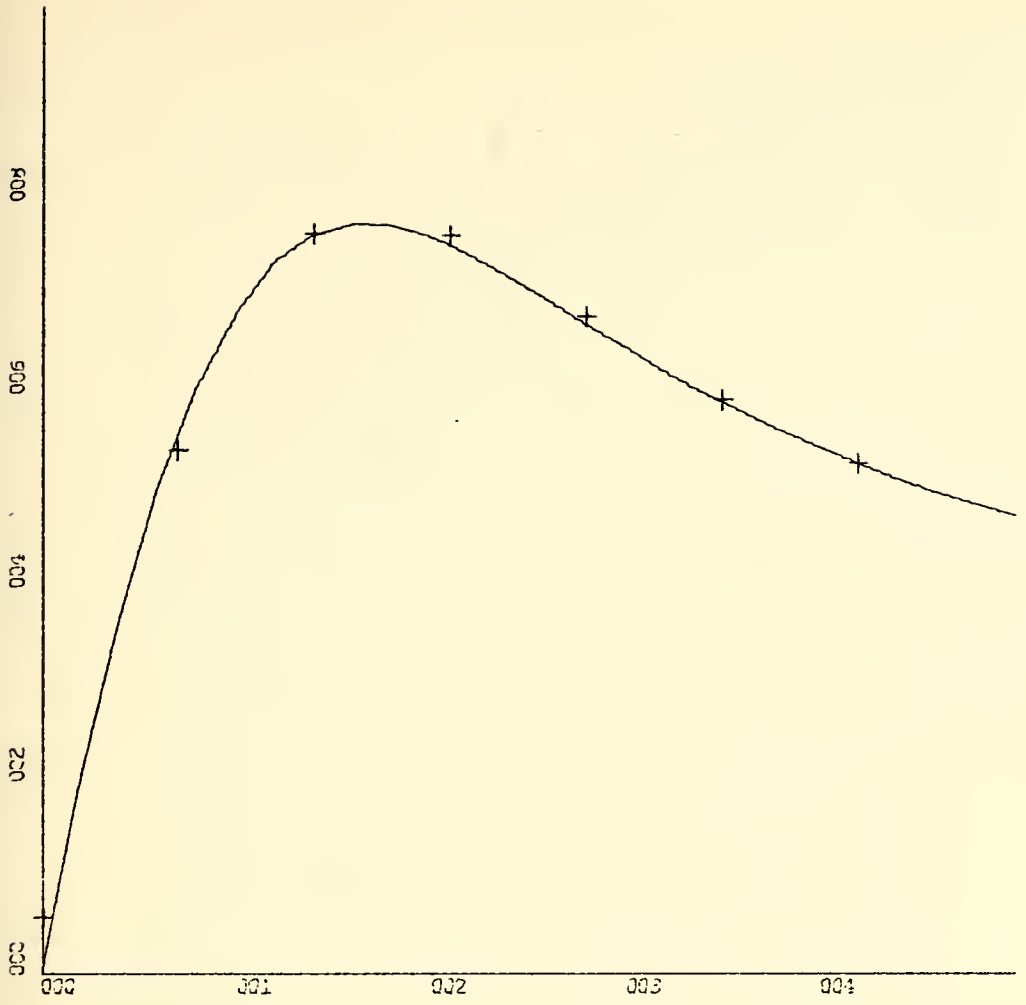
FIGURE 10



FOR DELTA T = 0.6 TFIN = 5.2

T	Q(N)
0.0	0.05000
0.6	0.48686
1.2	0.73288
1.8	0.77864
2.4	0.72725
3.0	0.65099
3.6	0.58165
4.2	0.52673

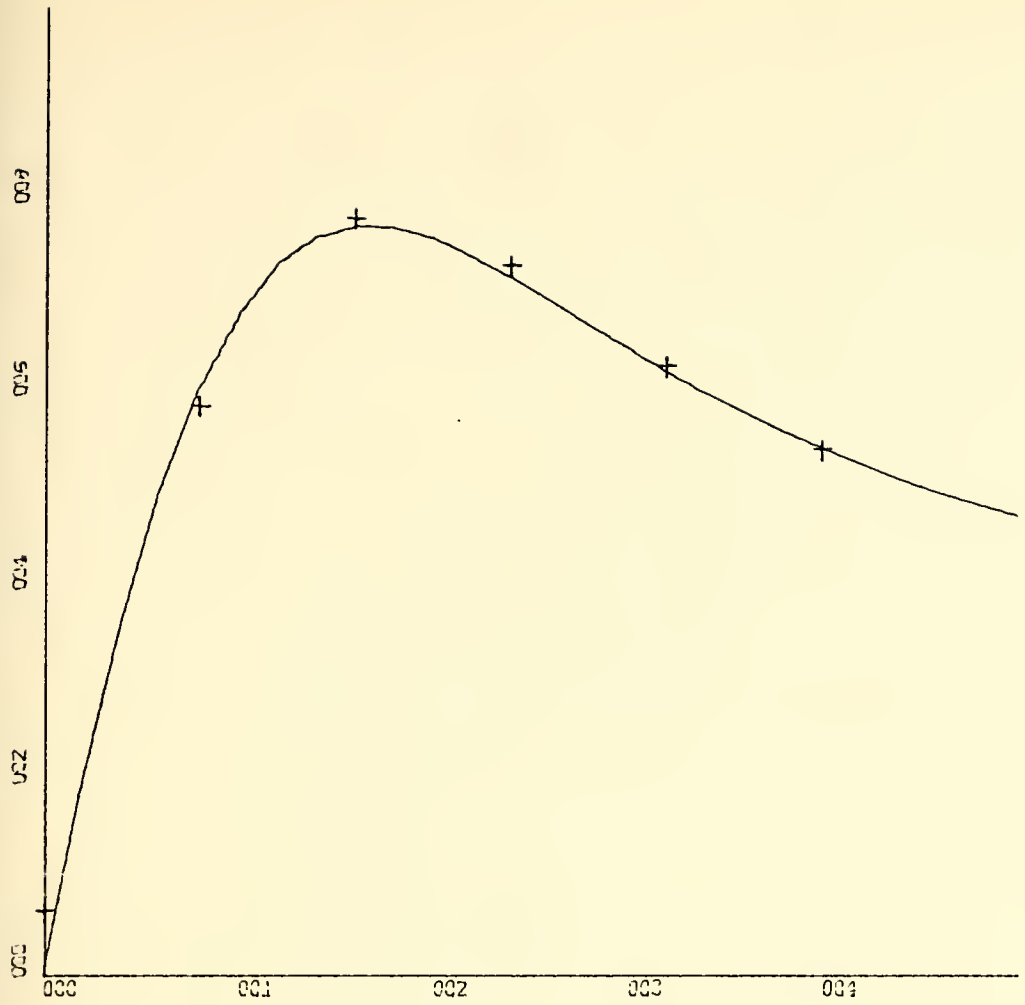
FIGURE 11



FOR DELTA T = 0.7 TFIN = 5.2

T	Q(N)
0.0	0.5833
0.7	0.54145
1.4	0.76577
2.1	0.76211
2.8	0.67831
4.2	0.52680

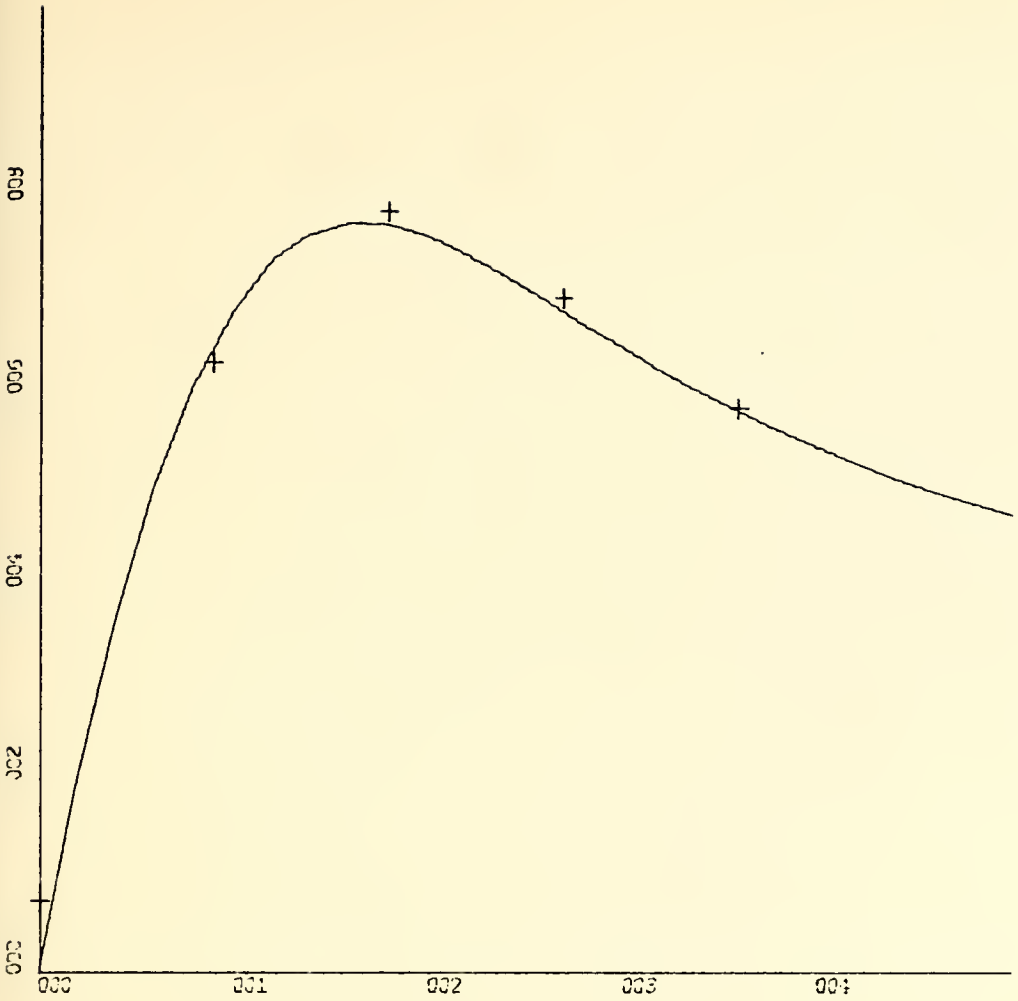
FIGURE 12



FOR DELTA T = 0.8 TFIN = 5.2

T	Q(N)
0.0	0.06667
0.8	0.58920
1.6	0.78249
2.4	0.73260
3.2	0.62874
4.0	0.54381

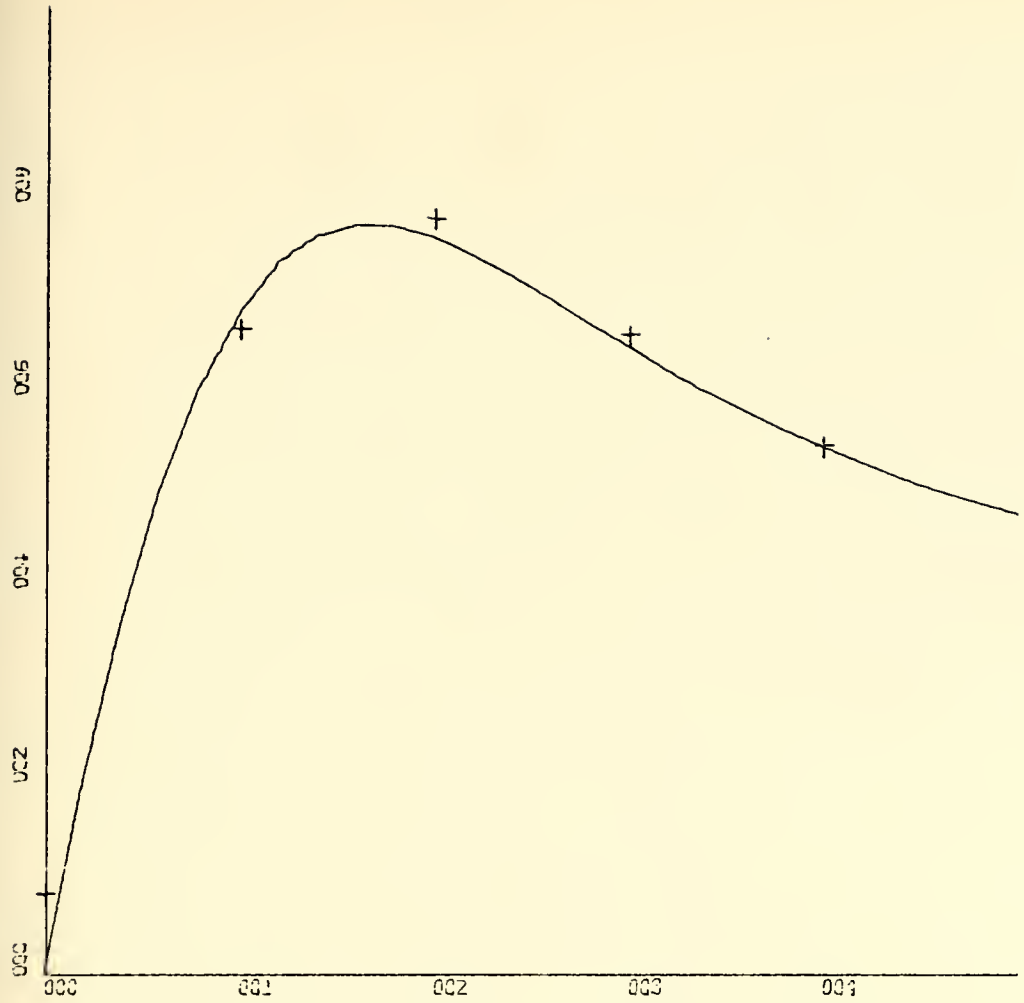
FIGURE 13



FOR DELTA T = 0.9 TFIN = 5.2

T	Q(N)
0.0	0.07500
0.9	0.63073
1.8	0.78658
2.7	0.69661
3.6	0.58340

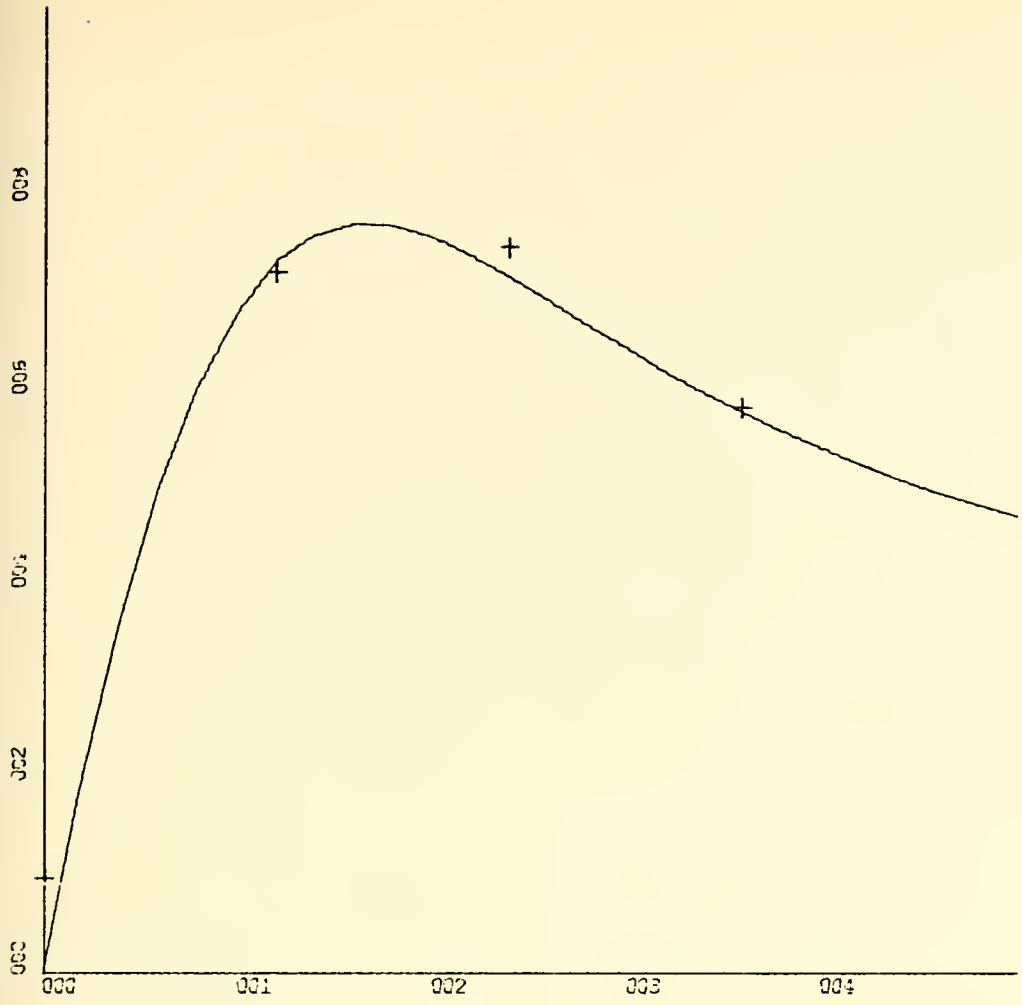
FIGURE 14



FOR DELTA T = 1.0 TFIN + 5.2

T	Q(N)
0.0	0.08333
1.0	0.66667
2.0	0.78105
3.0	0.65849
4.0	0.54411

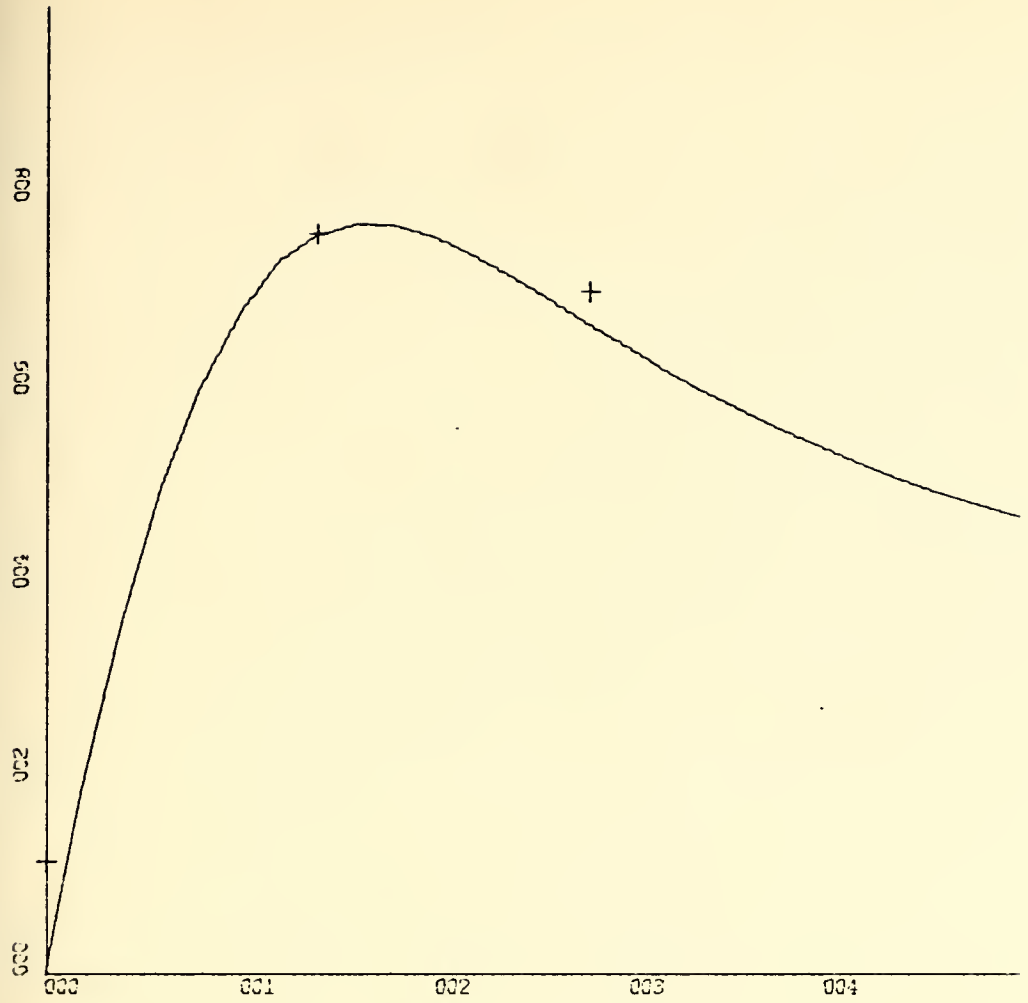
FIGURE 15



FOR DELTA T = 1.2 TFIN = 5.2

T	Q(N)
0.0	0.10000
1.2	0.72408
2.4	0.75053
3.6	0.58590

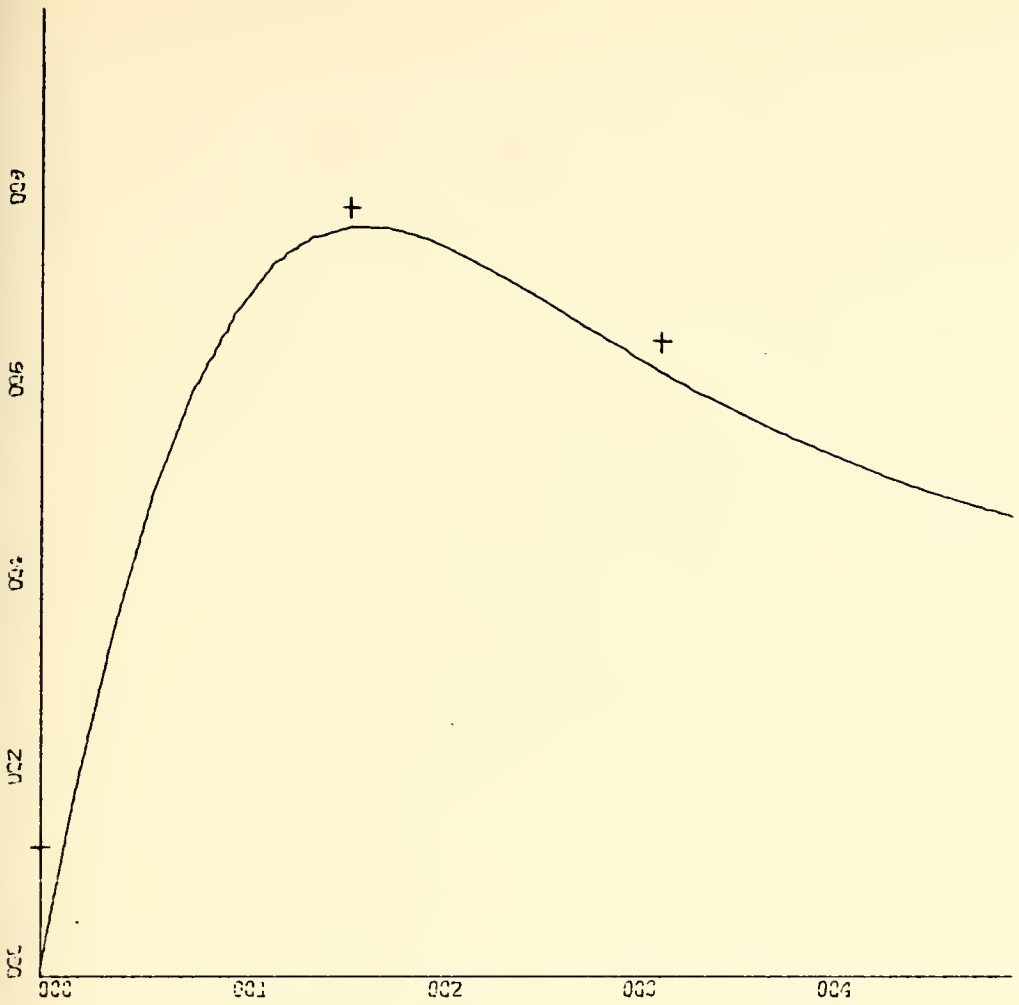
FIGURE 16



FOR DELTA T = 1.4 TFIN = 5.2

T	Q(N)
0.0	0.11667
1.4	0.76576
2.8	0.70534

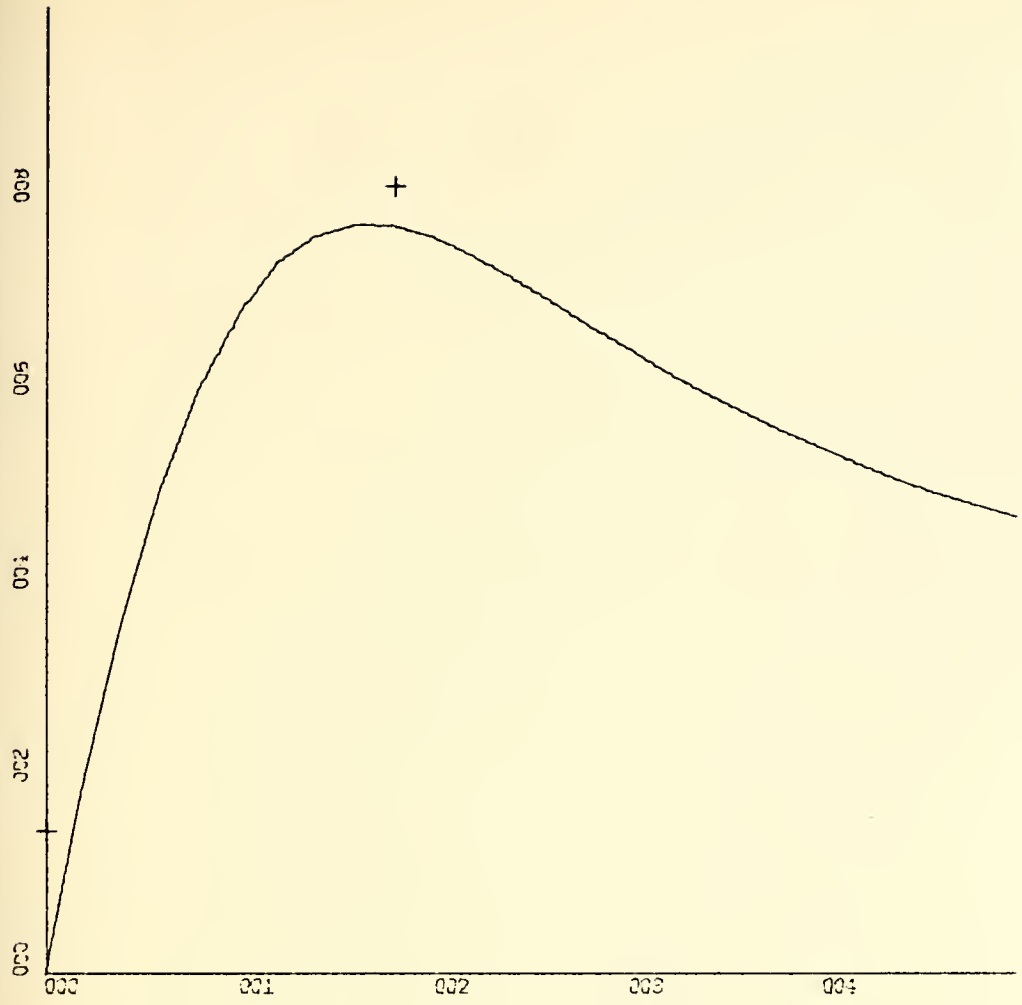
FIGURE 17



FOR DELTA T = 1.6 TFIN = 5.2

T	Q(N)
0.0	0.13333
1.6	0.79526
3.2	0.65424

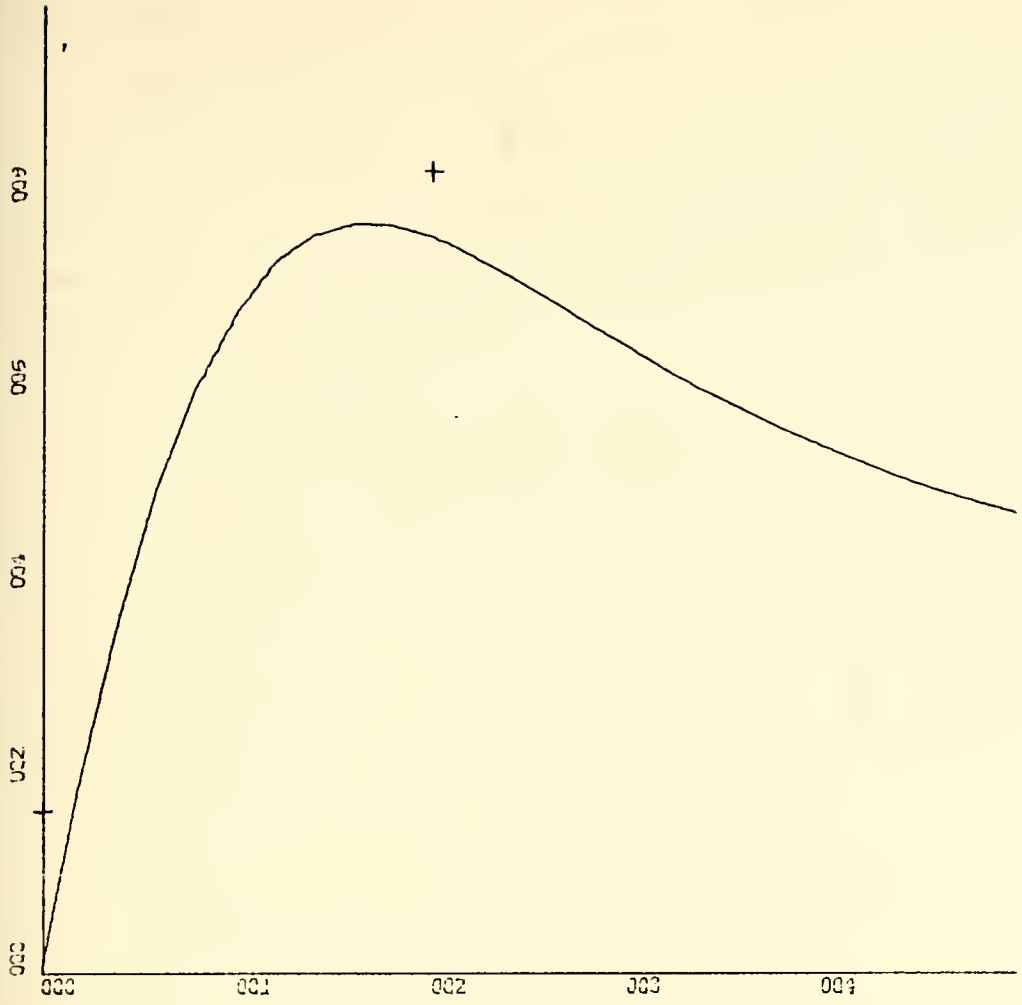
FIGURE 18



FOR DELTA T = 1.8 TFIN = 5.2

T	Q(N)
0.0	0.15000
1.8	0.81541

FIGURE 19



DELTA T = 2.0 TFIN = 5.2

T	Q(N)
0.0	0.16667
2.0	0.82843

FIGURE 20

1. Case 2 – $\dot{Y} + tY = 1$

The calculations are the same as in Case 1 with the only difference that P(t) is now equal to t. So, when dividing as in Equation (45), repeated here for convenience, the denominator into the numerator the substitution for each step is as P = t.

$$Y_A^*(z) = \frac{T + 10Tz^{-1} + Tz^{-2}}{(12 + 6PT) - 24z^{-1} + (12 - 6PT)z^{-2}}$$

In the same manner, numerical and graphical output for T from 0.2 sec. to 2.0 sec. (Figs. 21 to 34) follow. Program 2 (page 47) shows a computer algorithm for solution of this problem.

FOR DELTA T = 0.2 TFIN = 5.2

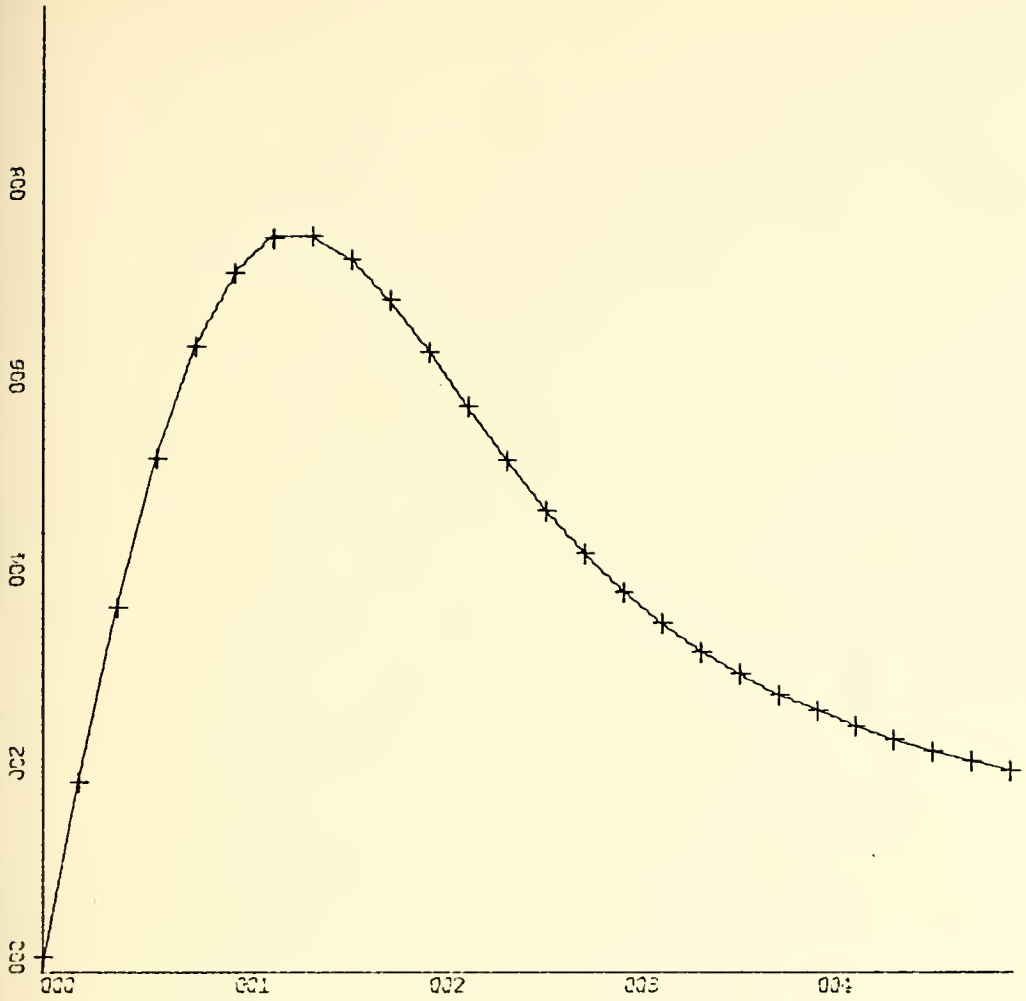
T	Q(N)	PR
0.0	0.01667	0.0
0.2	0.19608	0.19735
0.4	0.37707	0.37933
0.6	0.53018	0.53293
0.8	0.64664	0.64931
1.0	0.72264	0.72478
1.2	0.75927	0.76058
1.4	0.76154	0.76198
1.6	0.73700	0.73670
1.8	0.69413	0.69335
2.0	0.64099	0.64000
2.2	0.58425	0.58329
2.4	0.52880	0.52801
2.6	0.47769	0.47713
2.8	0.43241	0.43208
3.0	0.39333	0.39319
3.2	0.36010	0.36008
3.4	0.33199	0.33204
3.6	0.30817	0.30825
3.8	0.28784	0.28793

FOR DELTA T = 0.2 TFIN = 5.2
(Continued)

T	Q(N)	PR
4.0	0.27033	0.27041
4.2	0.25507	0.25514
4.4	0.24162	0.24168
4.6	0.22966	0.22971
4.8	0.21893	0.21897
5.0	0.29023	0.20926

```
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
C   Q(T)=STEP , P(T)=T
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      INTEGER *4ITB(12)/12*0/
      REAL *4RTB(28)/28*0.0/
      DIMENSION A(100), Q(100), X(100), XX(100), PR(100)
      ITB(3) = 5
      ITB(4) = 5
C
      DO 1 I=1,26
      READ (5,7) XX(I),PR(I)
    1 CONTINUE
C
    2 READ (5,7,END=4) TD,TFIN
      WRITE (6,5) TD,TFIN
      WRITE (6,6)
      M = TFIN/TD
      A(1) = TD
      A(2) = 10.0*TD
      A(3) = TD
      A(4) = 0.0
C
      DO 3 N=1,M
      T = (N-1)*TD
C
      P = T
C
      F1 = 12.0+6.0*(P*TD)
      F2 = -24.0
      F3 = 12.0-6.0*(P*TD)
C
C   Q(N) IS THE OUTPUT
C
      Q(N) = A(N)/F1
      A(N+1) = A(N+1)-(Q(N)*F2)
      A(N+2) = A(N+2)-(Q(N)*F3)
      A(N+3) = 0.0
      X(N) = T
      WRITE (6,7) T,Q(N)
    3 CONTINUE
C
      WRITE (6,8)
      WRITE (6,7) (XX(I),PR(I),I=1,26)
      ITB(1) = 1
      ITB(2) = 0
      ITB(12) = 1
      CALL DRAWP (26,XX,PR,ITB,RTB)
      ITB(1) = 3
      ITB(2) = 2
      CALL DRAWP (M,X,Q,ITB,RTB)
      GO TO 2
    4 STOP
C
    5 FORMAT ('1','FOR DELTA T=',F6.2,4X,' TFIN=',F4.1)
    6 FORMAT ('  T=',7X,' Q(N)=')
    7 FORMAT (2F10.5)
    8 FORMAT ('  XX=',7X,'    PR=')
      END
```

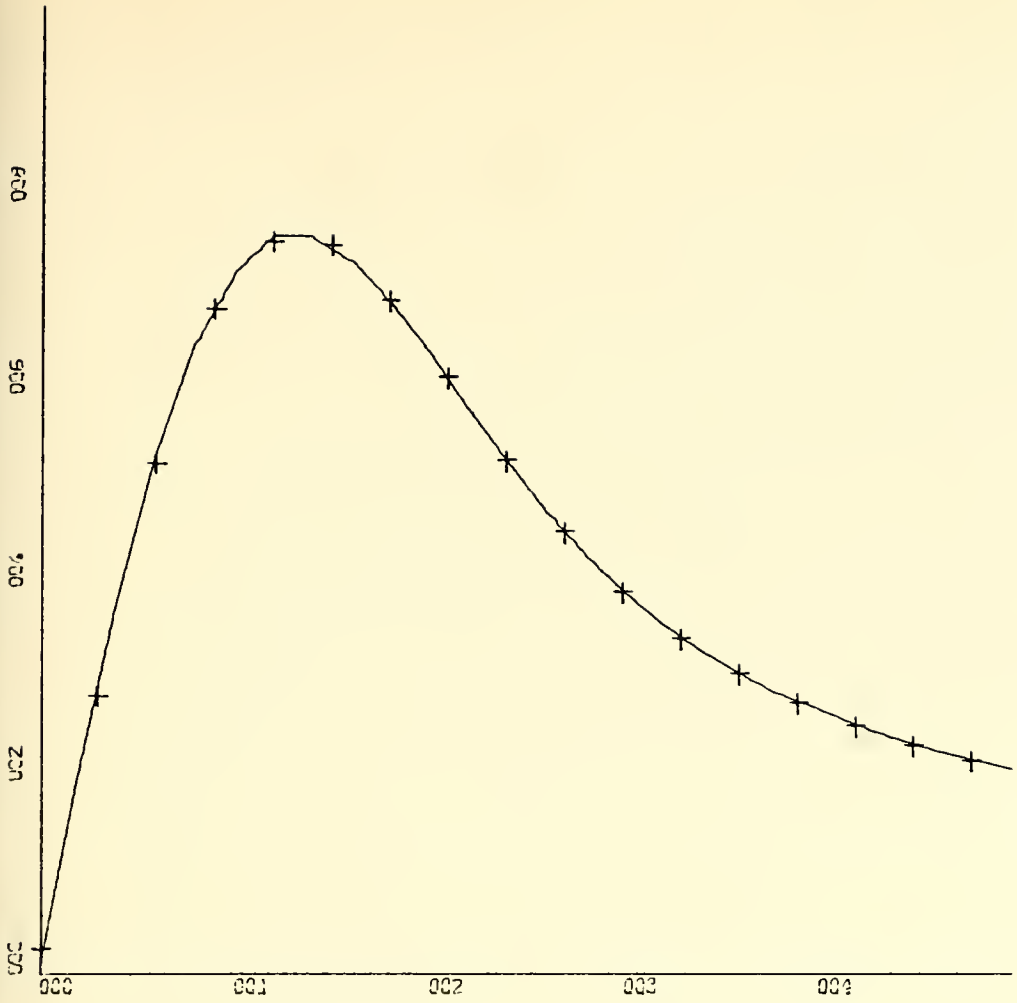
PROGRAM 2



x-scale = 1.0 units/inch
y-scale = 0.2 units/inch

FOR DELTA T = 0.2

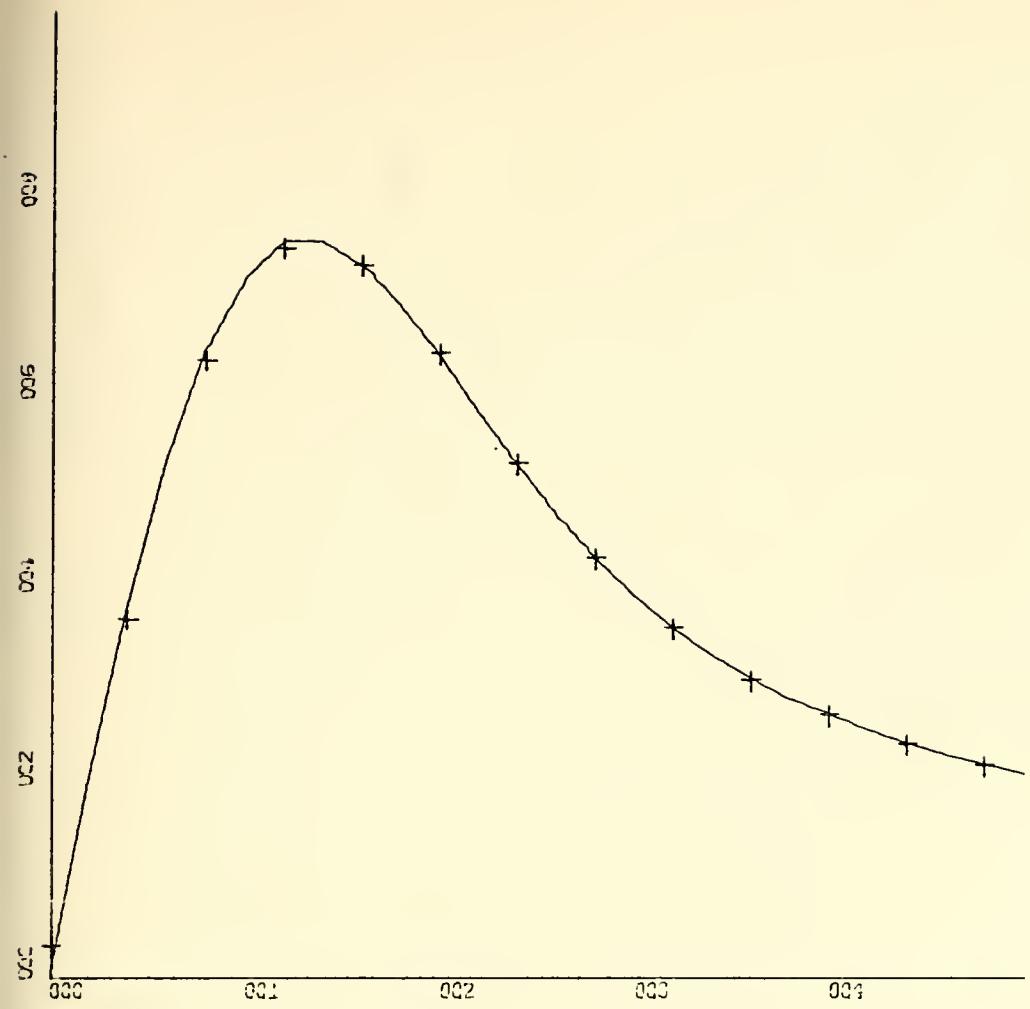
FIGURE 21



FOR DELTA T = 0.3 TFIN = 5.2

T	Q(N)
0.0	0.02500
0.3	0.28708
0.6	0.52675
0.9	0.68665
1.2	0.75759
1.5	0.75202
1.8	0.69513
2.1	0.61402
2.4	0.52986
2.7	0.45488
3.0	0.39355
3.3	0.34545
3.6	0.30809
3.9	0.27869
4.2	0.25500
4.5	0.23543
4.8	0.21891

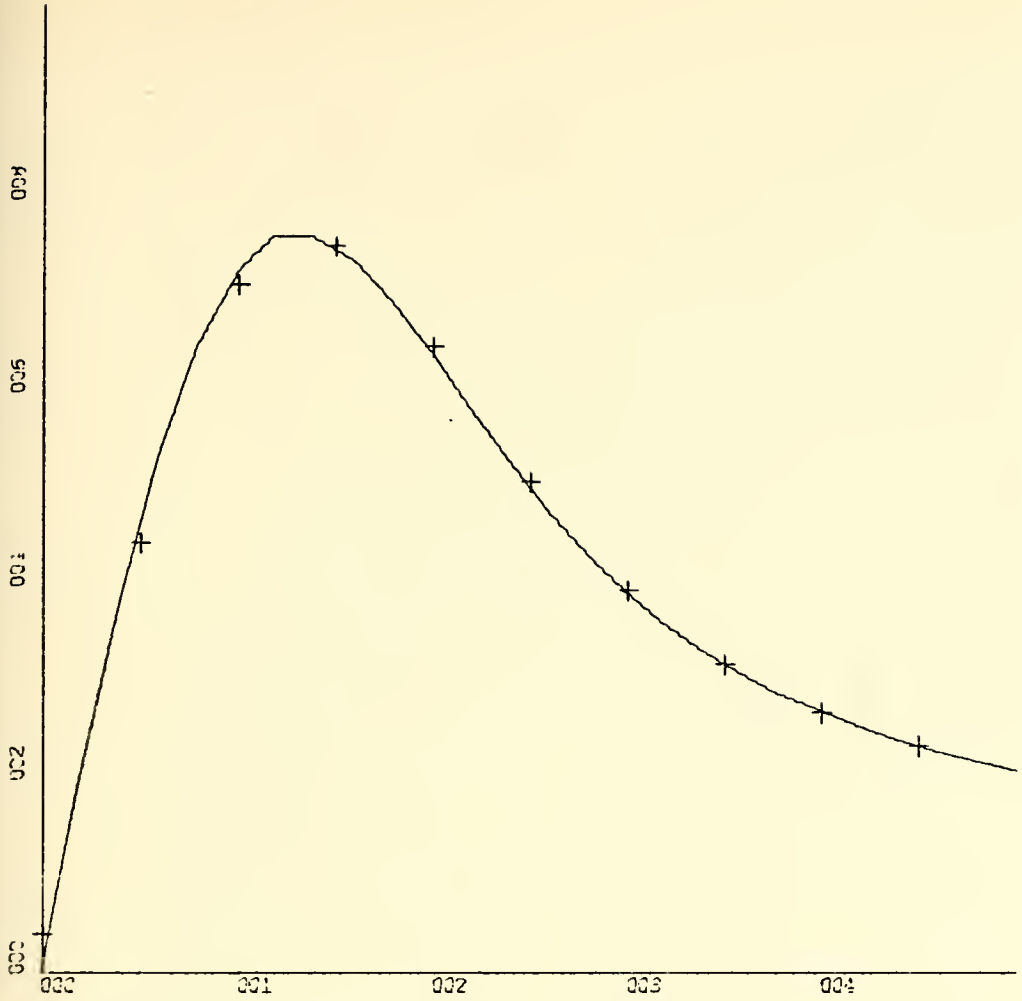
FIGURE 22



FOR DELTA T = 0.40 TFIN = 5.2

T	Q(N)
0.0	0.03333
0.4	0.37037
0.8	0.63857
1.2	0.75516
1.6	0.73782
2.0	0.64408
2.4	0.53138
2.8	0.43354
3.2	0.36022
3.6	0.30795
4.0	0.27013
4.4	0.24150
4.8	0.21887

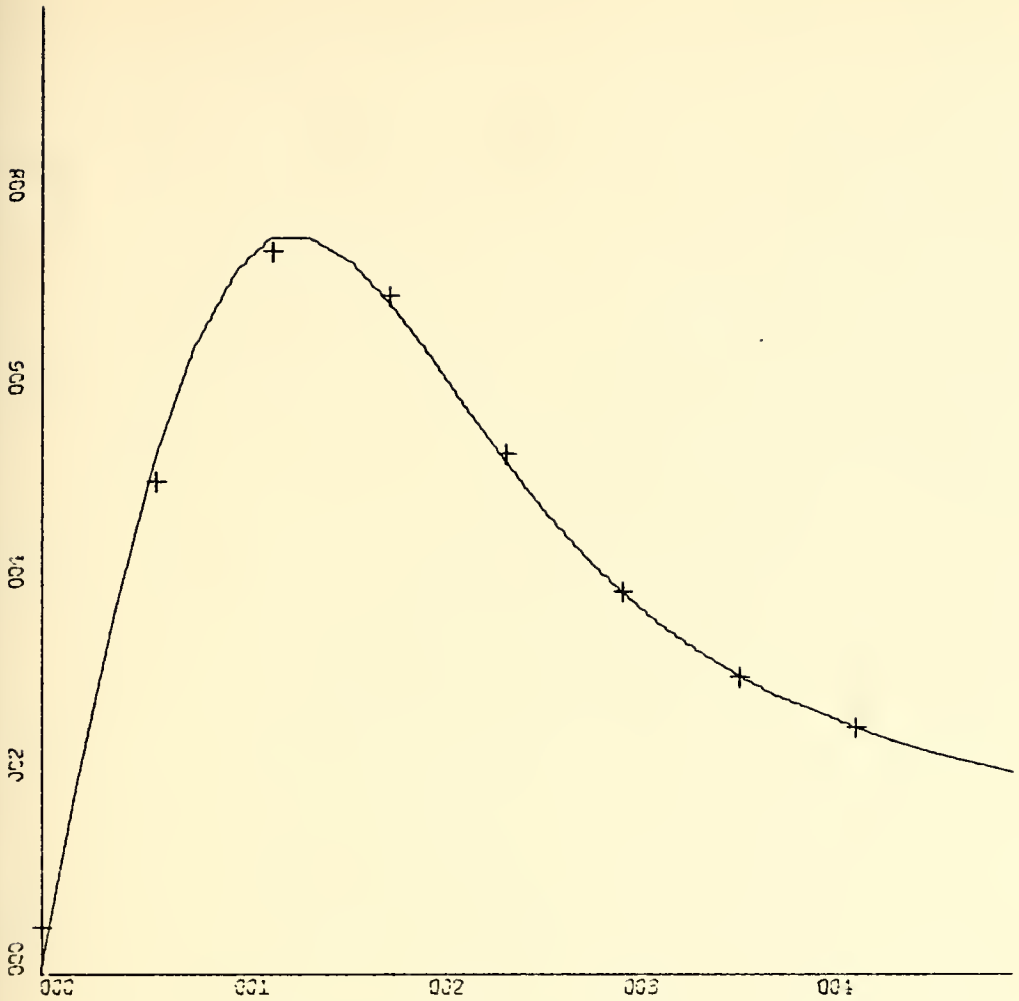
FIGURE 23



FOR DELTA T = 0.5 TFIN = 5.2

T	Q(N)
0.0	0.04167
0.5	0.44444
1.0	0.71111
1.5	0.75151
2.0	0.64646
2.5	0.50660
3.0	0.39427
3.5	0.31923
4.0	0.26995
4.5	0.23529

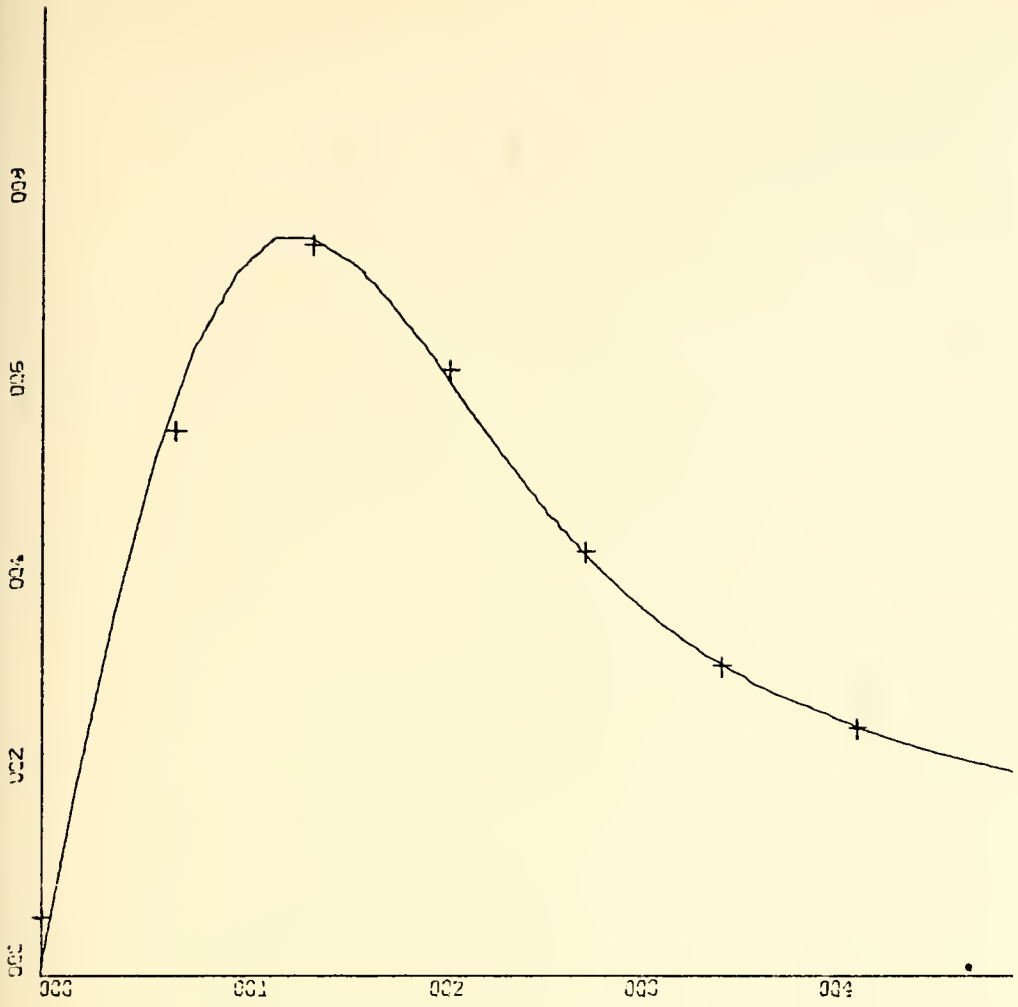
FIGURE 24



FOR DELTA T = 0.6 TFIN = 5.2

T	Q(N)
0.0	0.5000
0.6	0.50847
1.2	0.74776
1.8	0.70037
2.4	0.53615
3.0	0.39480
3.6	0.30744
4.2	0.25460

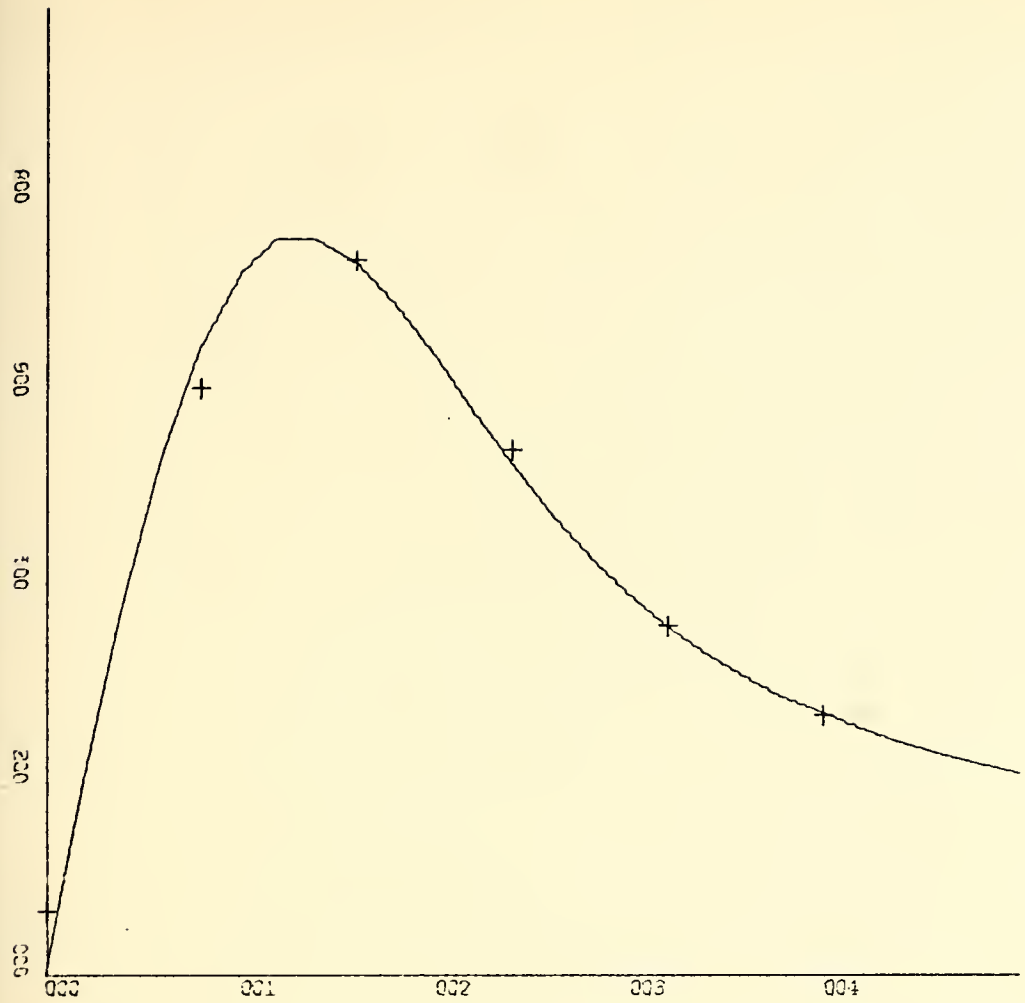
FIGURE 25



FOR DELTA T = 0.7 TFIN = 5.2

T	Q(N)
0.0	0.05833
0.7	0.56225
1.4	0.75470
2.1	0.62530
2.8	0.43722
3.5	0.31854
4.2	0.25438

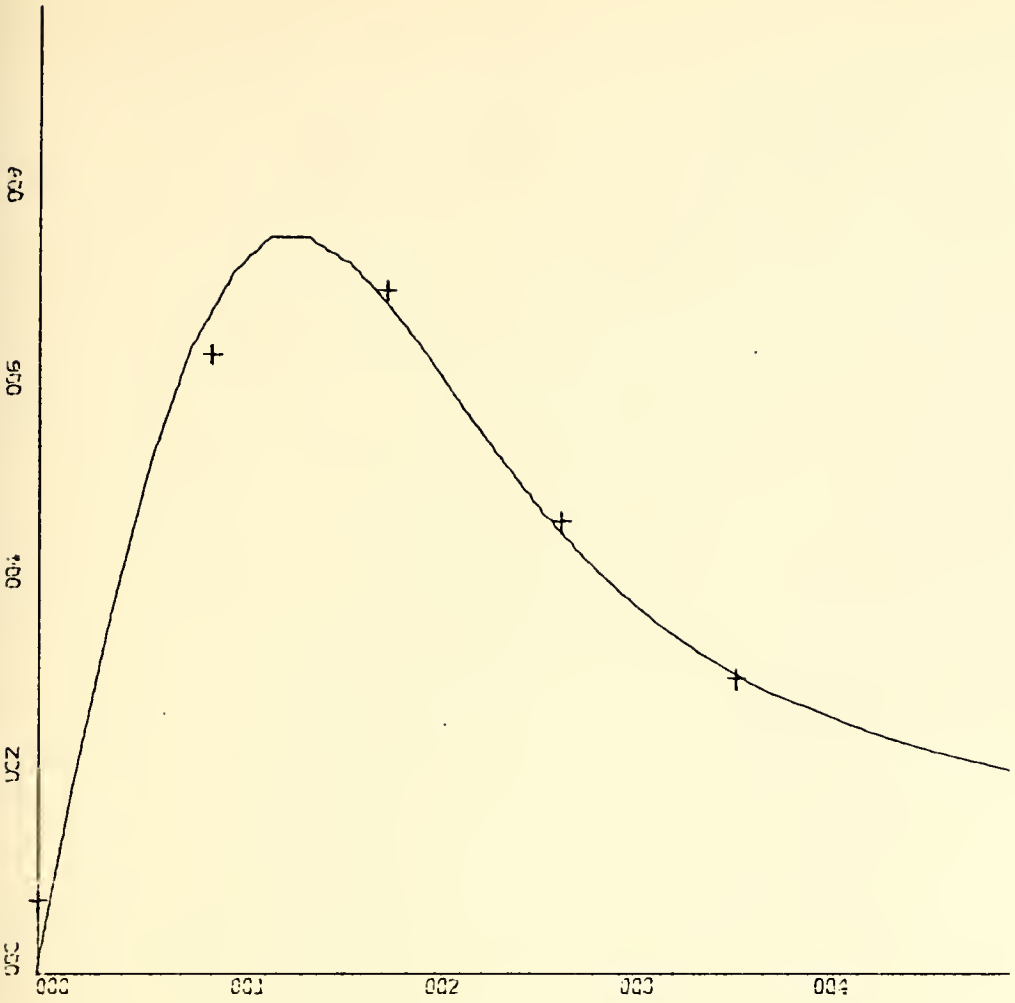
FIGURE 26



FOR DELTA T = 0.8 TFIN = 5.2

T	Q(N)
0.0	0.06667
0.8	0.60606
1.6	0.73910
2.4	0.54392
3.2	0.36042
4.0	0.26888

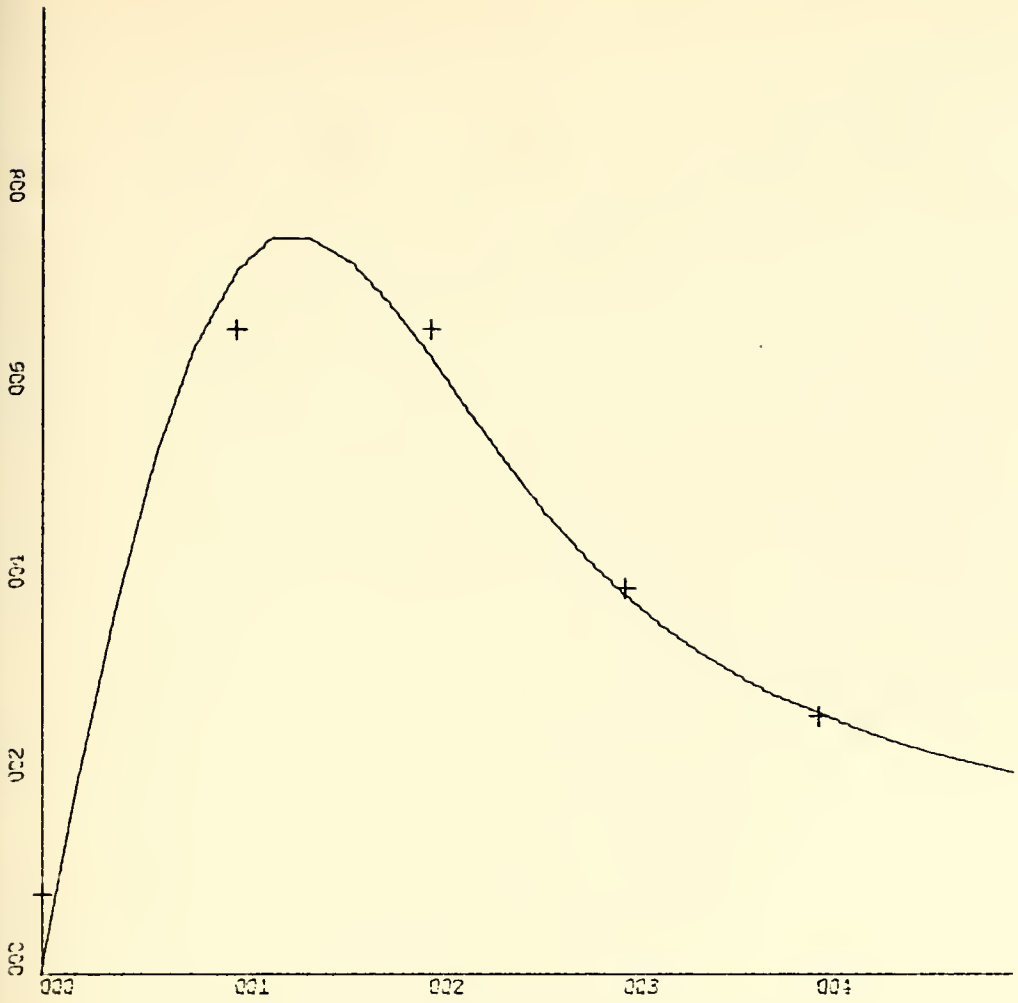
FIGURE 27



FOR DELTA T = 0.9 TFIN = 5.2

T	Q(N)
0.0	0.07500
0.9	0.64057
1.8	0.70781
2.7	0.46704
3.6	0.30519

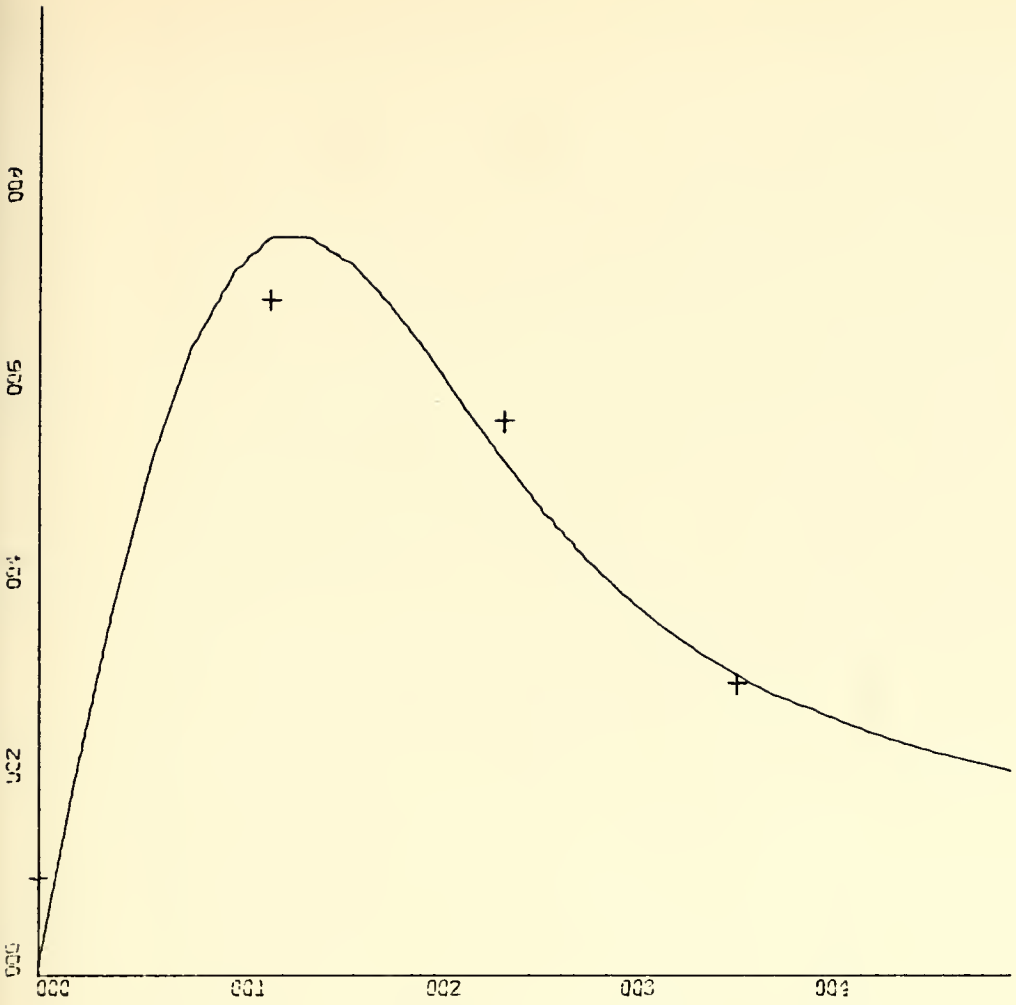
FIGURE 28



FOR DELTA T = 1.0 TFIN = 5.2

T	Q(N)
0.0	0.08333
1.0	0.66667
2.0	0.66667
3.0	0.40000
4.0	0.26667

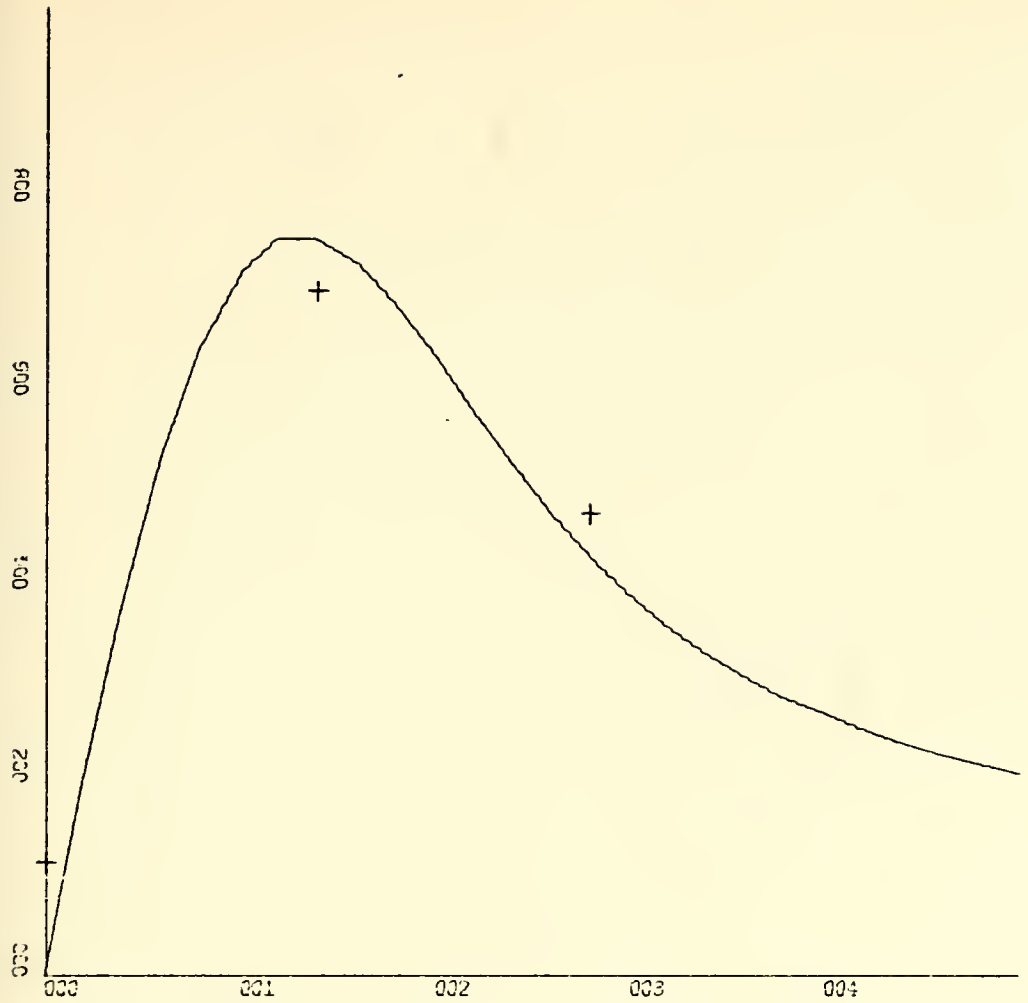
FIGURE 29



FOR DELTA T = 1.2 TFIN = 5.2

T	Q(N)
0.0	0.10000
1.2	0.69767
2.4	0.57186
3.6	0.30012

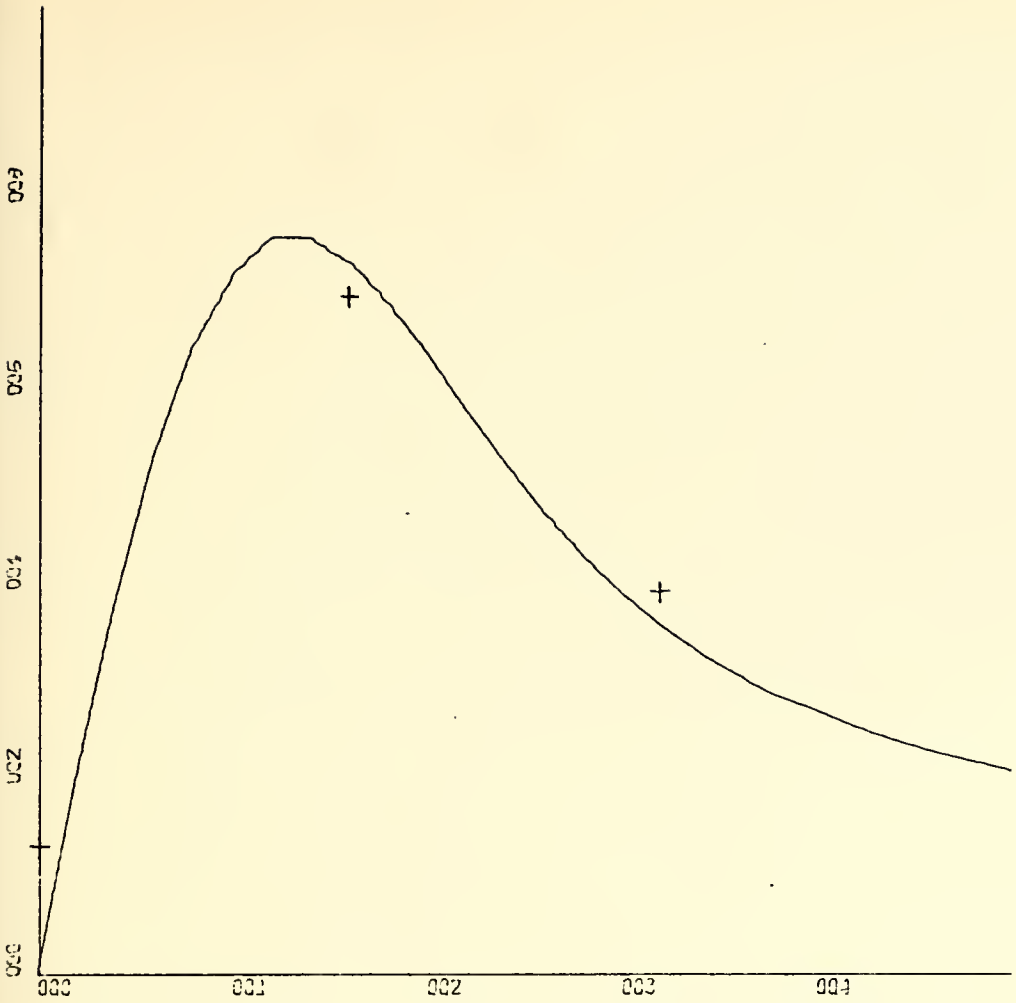
FIGURE 30



FOR DELTA T = 1.4 TFIN = 5.2

T	Q(N)
0.0	0.11667
1.4	0.70707
2.8	0.47775

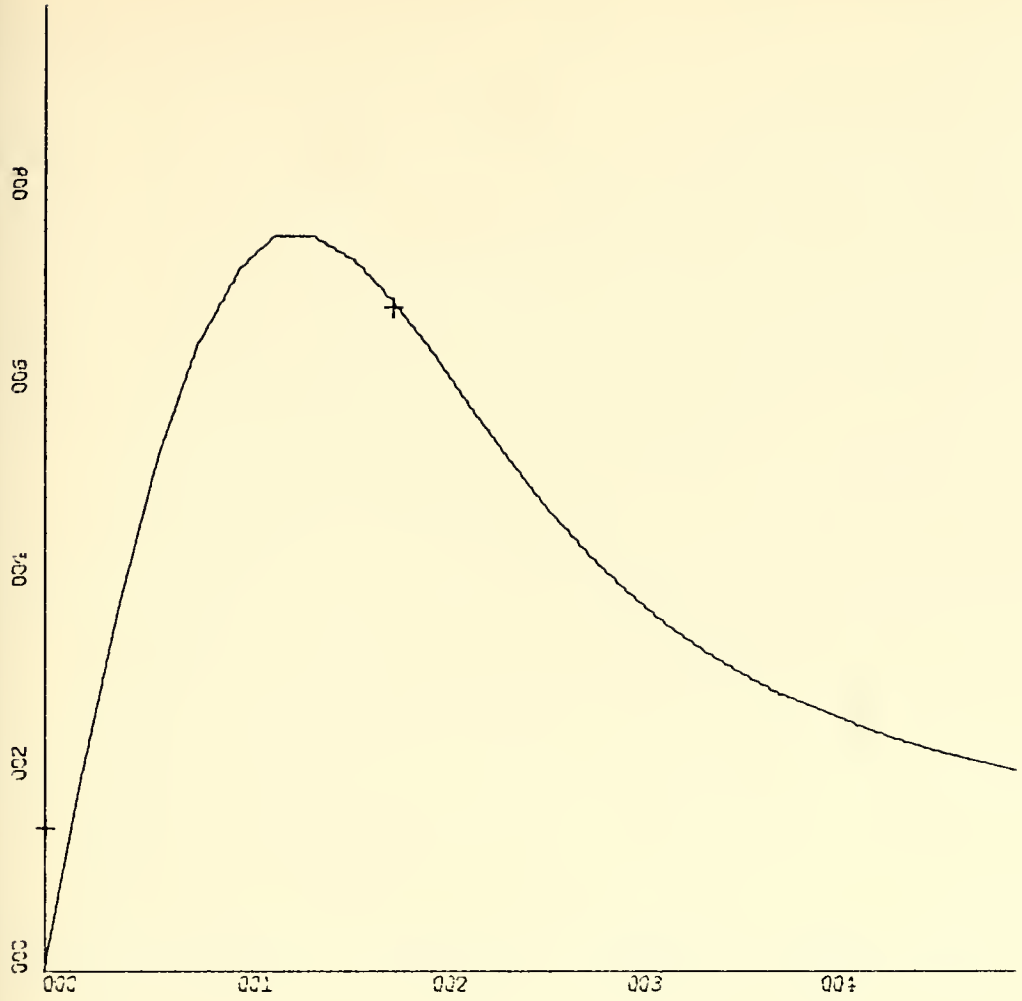
FIGURE 31



FOR DELTA T = 1.6 TFIN = 5.2

T	Q(N)
0.0	0.13333
1.6	0.70175
3.2	0.39424

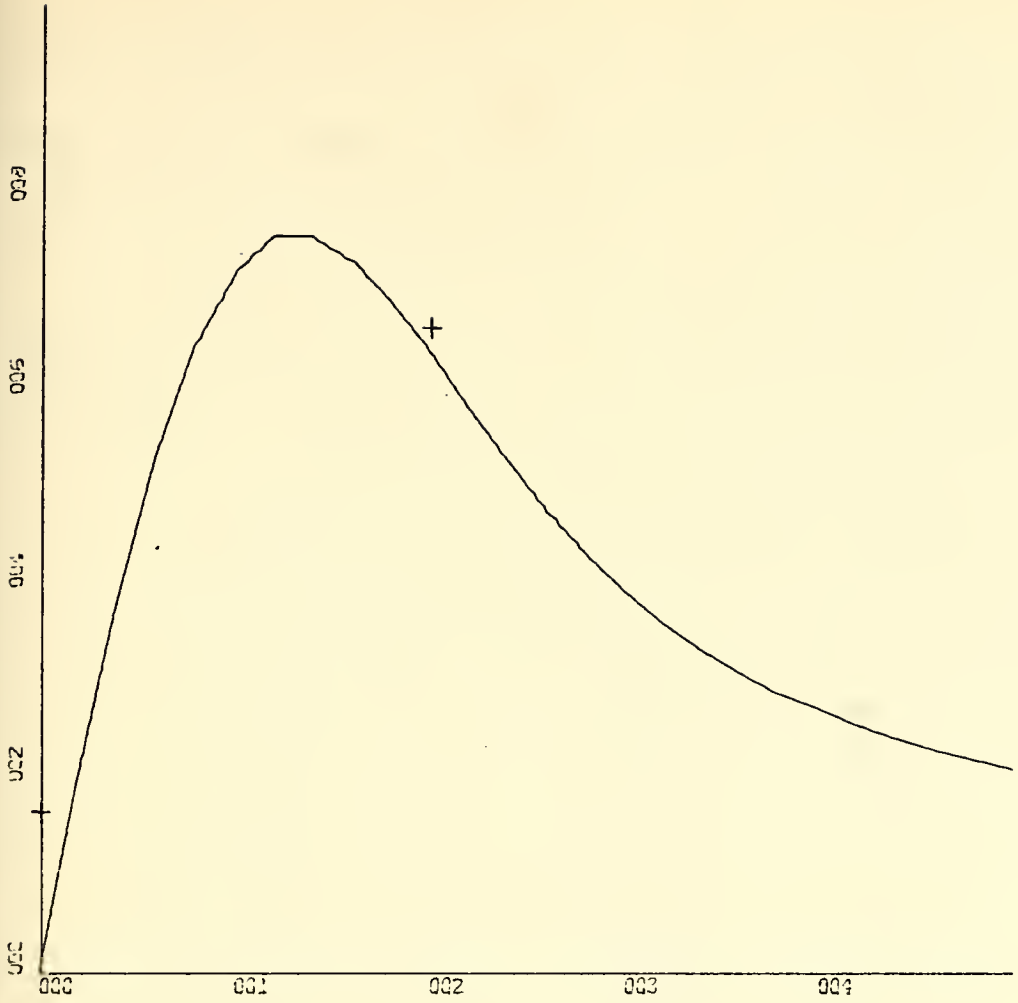
FIGURE 32



FOR DELTA T = 1.8 TFIN = 5.2

T	Q(N)
0.0	0.15000
1.8	0.68702

FIGURE 33



FOR DELTA T = 2.0 TFIN = 5.2

T	Q(N)
0.0	0.16667
2.0	0.66667

FIGURE 34

2. Case 3 — $\dot{Y} + t^2 Y = 1$

As in Case 1 and Case 2, the same calculations are followed and only in the division, the substitution of $P(t) = t^2$ is the sole difference.

Output for several time iterations follows as in the previous cases (Figs. 35 to 48). Computer algorithm is shown in Program 3 (page 63).

FOR DELTA T = 0.2 TFIN = 5.2

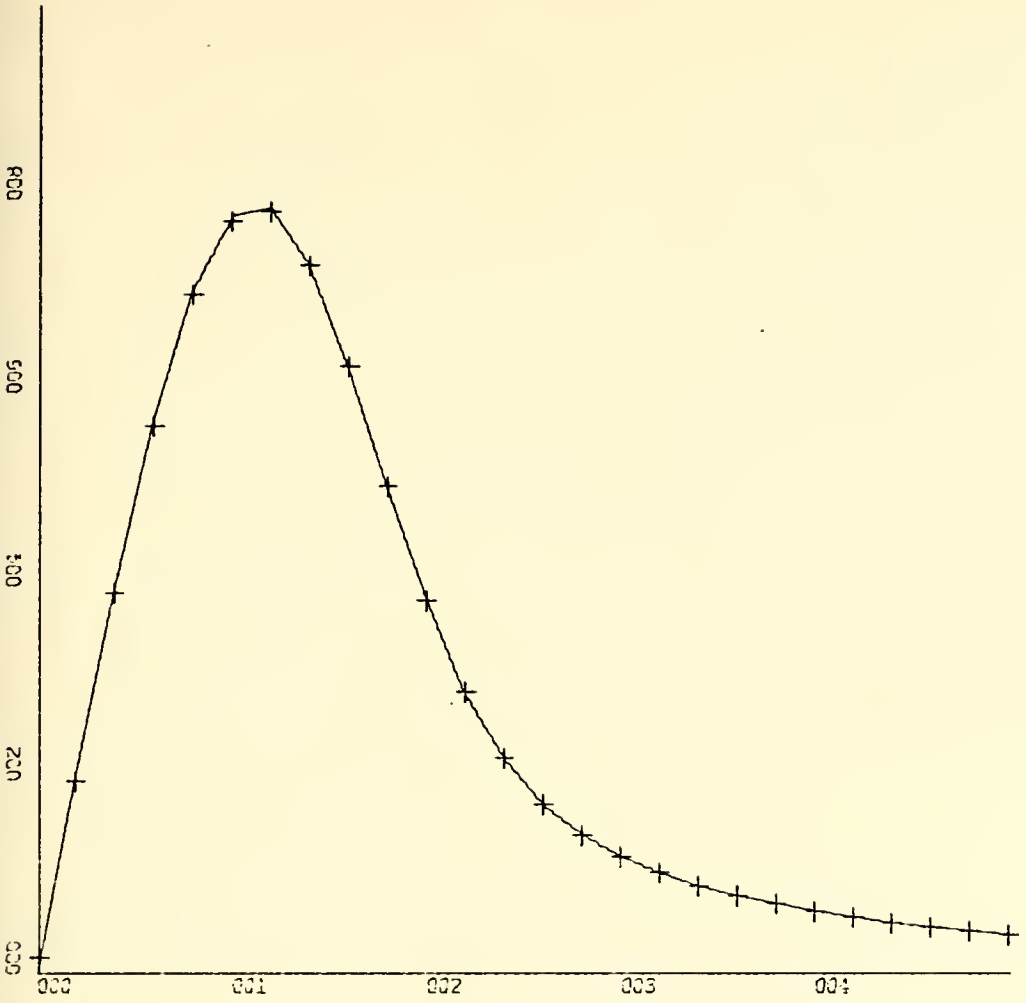
T	Q(N)	PR
0.0	0.01667	0.0
0.2	0.19920	0.19960
0.4	0.39213	0.39366
0.6	0.56550	0.56858
0.8	0.70032	0.70472
1.0	0.77773	0.78240
1.2	0.78667	0.79011
1.4	0.73026	0.73132
1.6	0.62669	0.62542
1.8	0.50321	0.50080
2.0	0.38584	0.38376
2.2	0.29077	0.28980
2.4	0.22210	0.22208
2.6	0.17552	0.17589
2.8	0.14398	0.14433
3.0	0.12163	0.12183
3.2	0.10482	0.10492
3.4	0.09160	0.09165
3.6	0.08088	0.08091
3.8	0.07204	0.07205
4.0	0.06462	0.06464
4.2	0.05833	0.05834
4.4	0.05294	0.05295
4.6	0.04828	0.04830
4.8	0.04422	0.04423
5.0	0.04067	0.04067

Equation (45) is repeated here for convenience:

$$y_A^*(z) = \frac{T + 10Tz^{-1} + Tz^{-2}}{(12 + 6PT) - 24z^{-1} + (12 - 6PT)z^{-2}}$$

```
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
C  Q(T)=STEP , P(T)=T**2
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      INTEGER *4ITB(12)/12*0/
      REAL *4RTB(28)/28*0.0/
      DIMENSION A(100), Q(100), X(100), XX(100), PR(100)
      ITB(3) = 5
      ITB(4) = 5
C
      DO 1 I=1,26
      READ (5,7) XX(I),PR(I)
    1 CONTINUE
C
    2 READ (5,7,END=4) TD,TFIN
      WRITE (6,5) TD,TFIN
      WRITE (6,6)
      M = TFIN/TD
      A(1) = TD
      A(2) = 10.0*TD
      A(3) = TD
      A(4) = 0.0
C
      DO 3 N=1,M
      T = (N-1)*TD
C
      P = T**2
C
      F1 = 12.0+6.0*(P*TD)
      F2 = -24.0
      F3 = 12.0-6.0*(P*TD)
C
C     Q(N) IS THE OUTPUT
C
      Q(N) = A(N)/F1
      A(N+1) = A(N+1)-(Q(N)*F2)
      A(N+2) = A(N+2)-(Q(N)*F3)
      A(N+3) = 0.0
      X(N) = T
      WRITE (6,7) T,Q(N)
    3 CONTINUE
C
      WRITE (6,8)
      WRITE (6,7) (XX(I),PR(I),I=1,26)
      ITB(1) = 1
      ITB(2) = 0
      ITB(12) = 1
      CALL DRAWP (26,XX,PR,ITB,RTB)
      ITB(1) = 3
      ITB(2) = 2
      CALL DRAWP (M,X,Q,ITB,RTB)
      GO TO 2
    4 STOP
C
    5 FORMAT ('1','FOR DELTA T=',F6.2,4X,' TFIN=',F4.1)
    6 FORMAT ('   T=',7X,' Q(N)=')
    7 FORMAT (2F10.5)
    8 FORMAT ('  XX=',7X,'    PR=')
      END
```

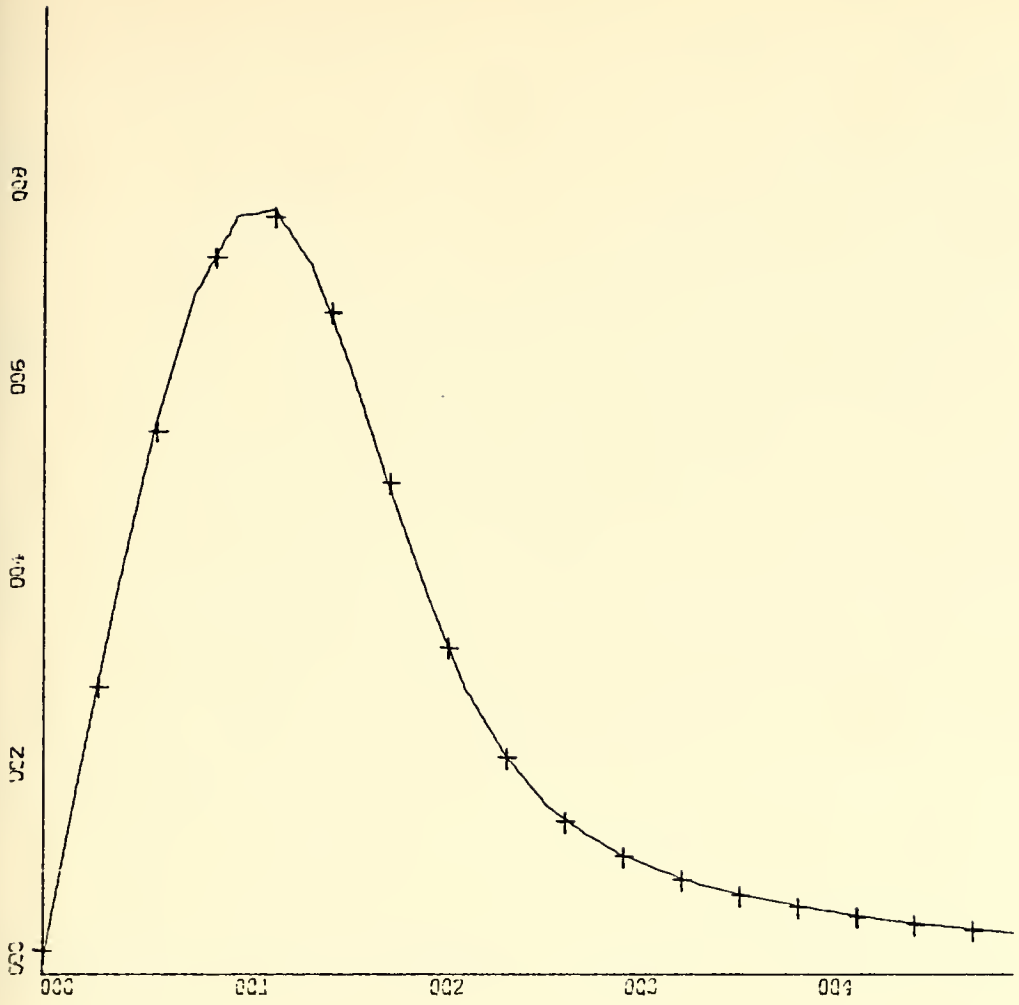
PROGRAM 3



x-scale = 1.0 units/inch
y-scale = 0.2 units/inch

FOR DELTA T = 0.2

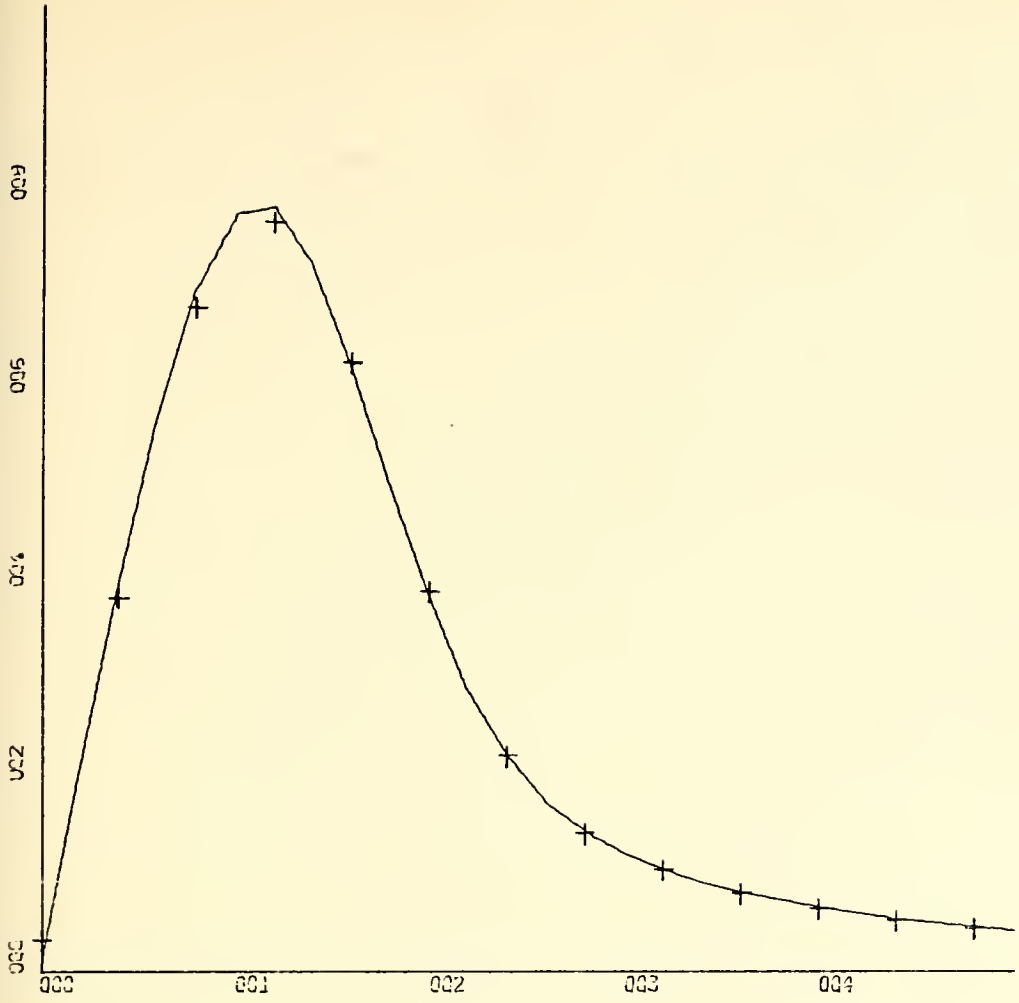
FIGURE 35



FOR DELTA T = 0.3 TFIN = 5.2

T	Q(N)
0.0	0.02500
0.3	0.29600
0.6	0.56168
0.9	0.74128
1.2	0.78225
1.5	0.68283
1.8	0.50631
2.1	0.33719
2.4	0.22218
2.7	0.15773
3.0	0.12138
3.3	0.09778
3.6	0.08086
3.9	0.06816
4.2	0.05832
4.5	0.05053
4.8	0.04422

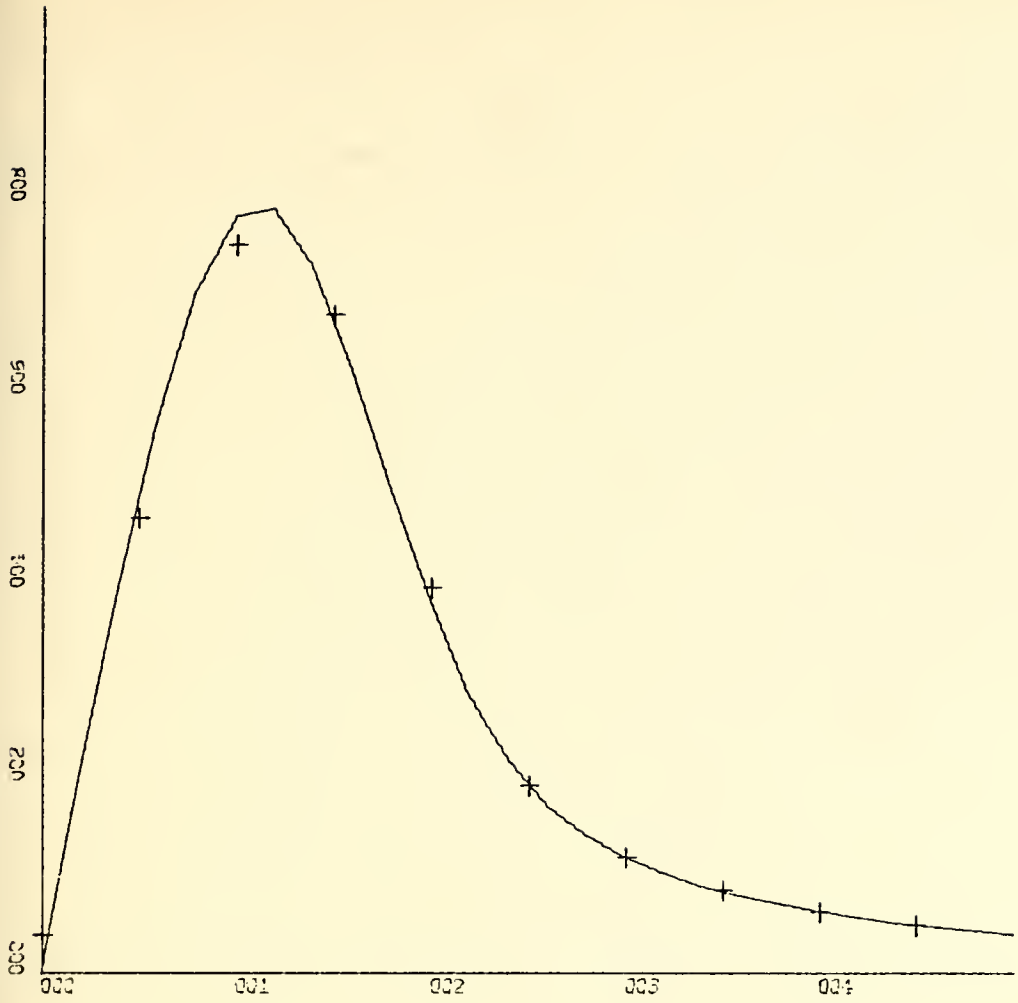
FIGURE 36



FOR DELTA T = 0.4 TFIN = 5.2

T	Q(N)
0.0	0.03333
0.4	0.38760
0.8	0.68723
1.2	0.77582
1.6	0.62989
2.0	0.39299
2.4	0.22240
2.8	0.14260
3.2	0.10466
3.6	0.08082
4.0	0.06460
4.4	0.05293
4.8	0.04422

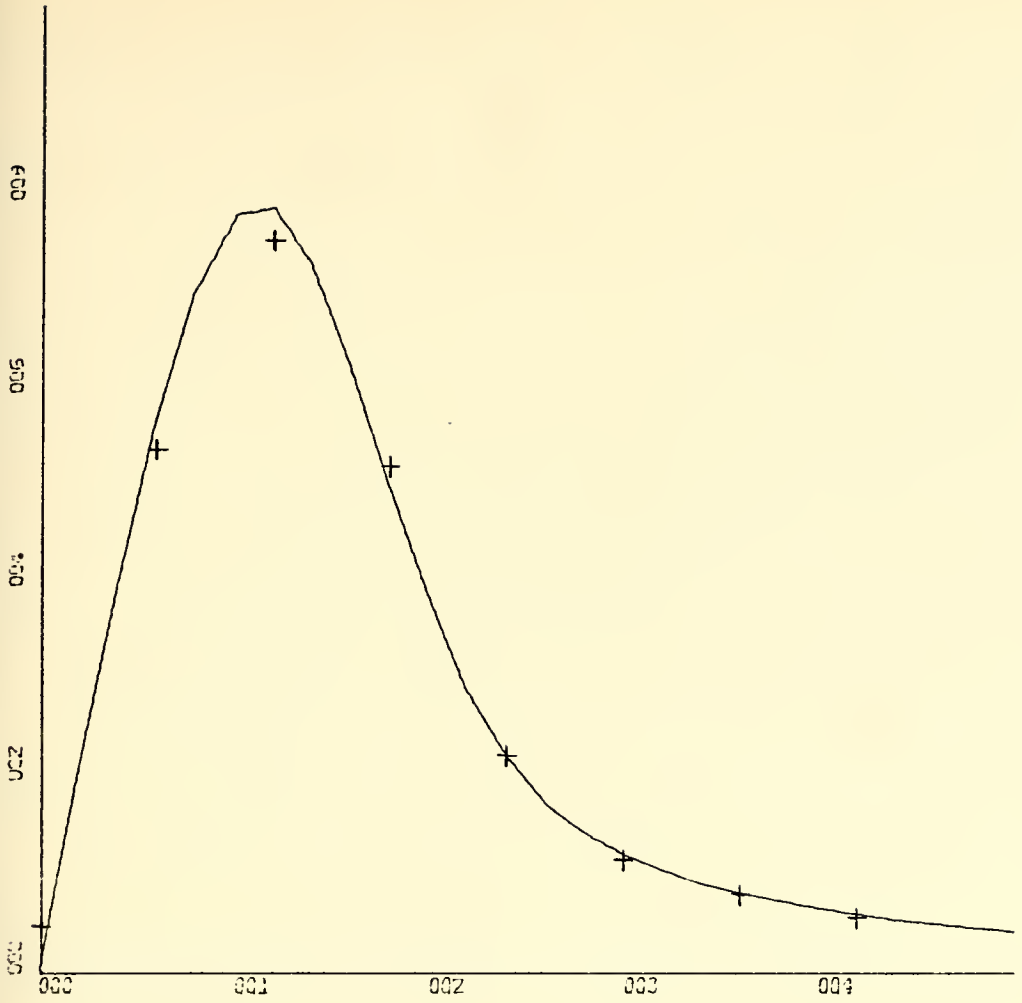
FIGURE 37



FOR DELTA T = 0.5 TFIN = 5.2

T	Q(N)
0.0	0.04167
0.5	0.47059
1.0	0.75294
1.5	0.68141
2.0	0.39906
2.5	0.19512
3.0	0.12007
3.5	0.08613
4.0	0.06447
4.5	0.05057

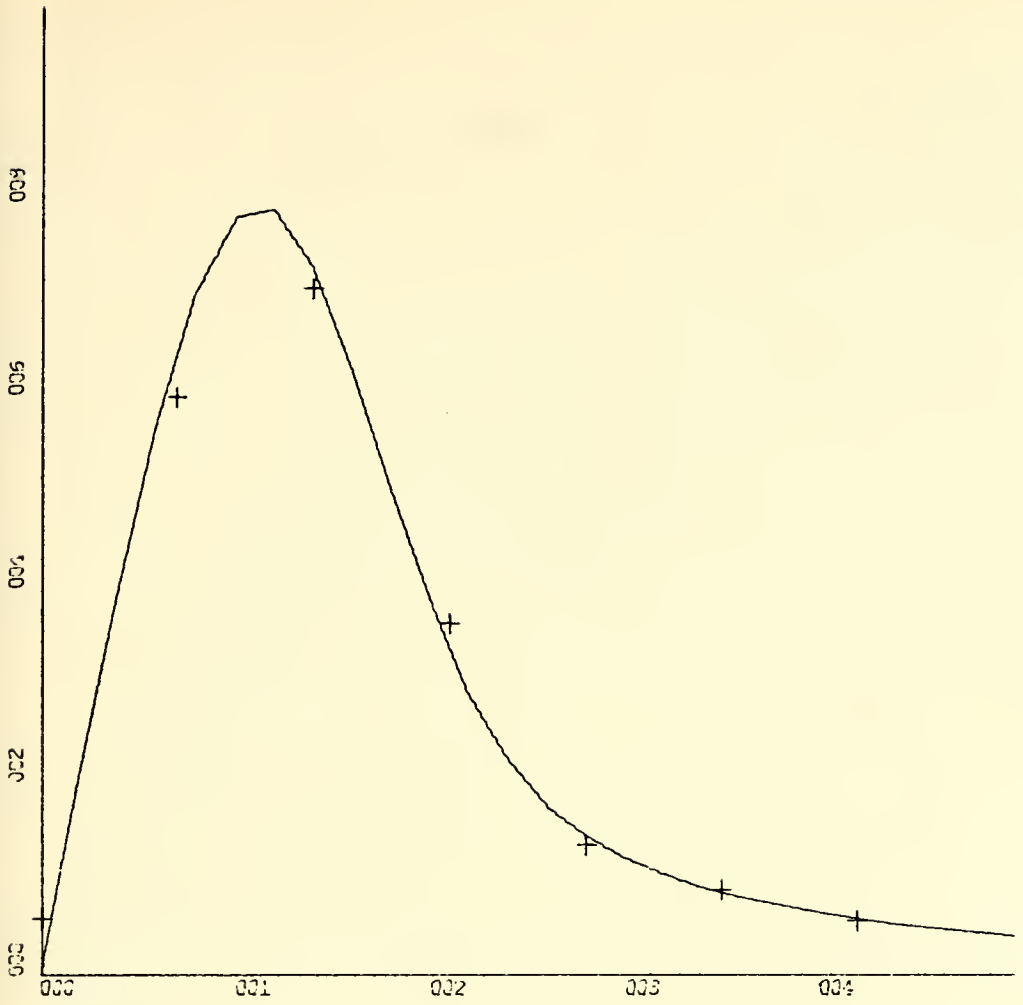
FIGURE 38



FOR DELTA T = 0.6 TFIN = 5.2

T	Q(N)
0.0	0.05
0.6	0.54152
1.2	0.75631
1.8	0.52210
2.4	0.22530
3.0	0.11783
3.6	0.08177
4.2	0.05783

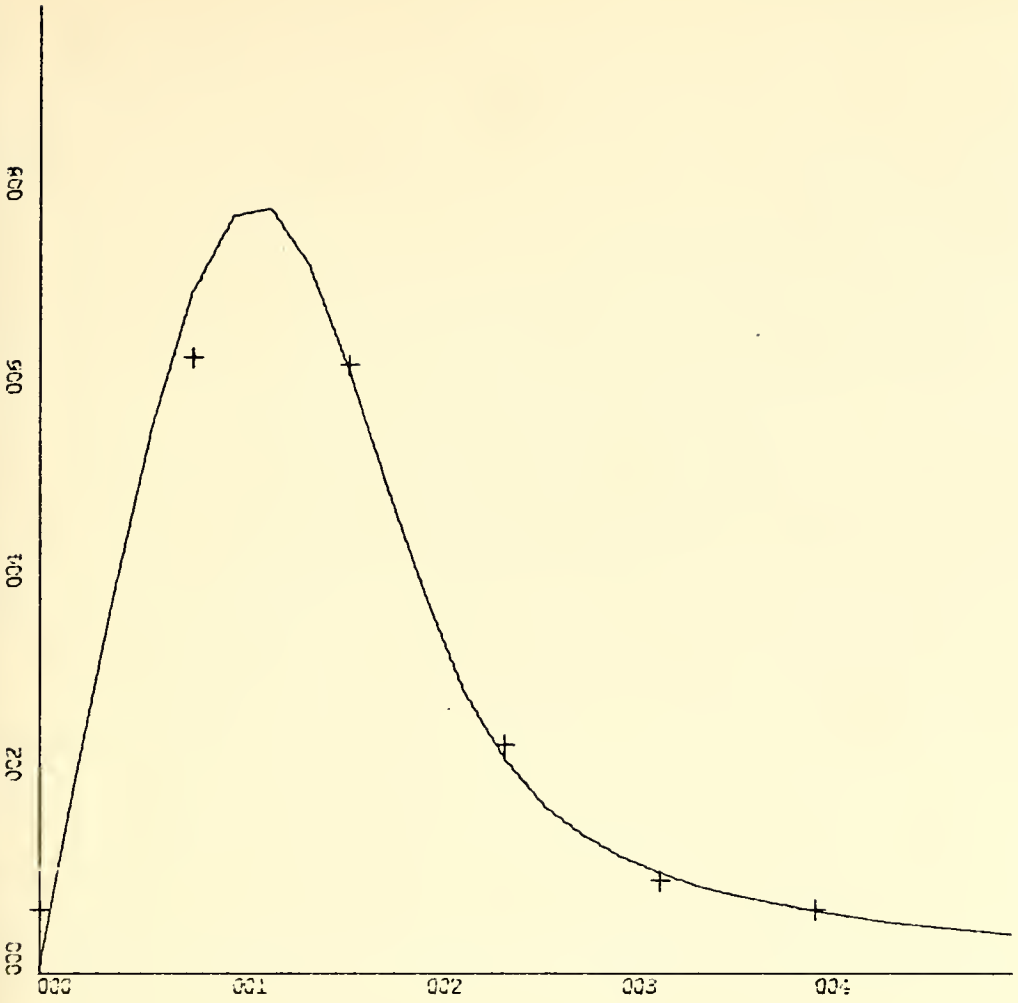
FIGURE 39



FOR DELTA T = 0.7 TFIN = 5.2

T	Q(N)
0.0	0.05833
0.7	0.59752
1.4	0.70881
2.1	0.36272
2.8	0.13431
3.5	0.08809
4.2	0.05721

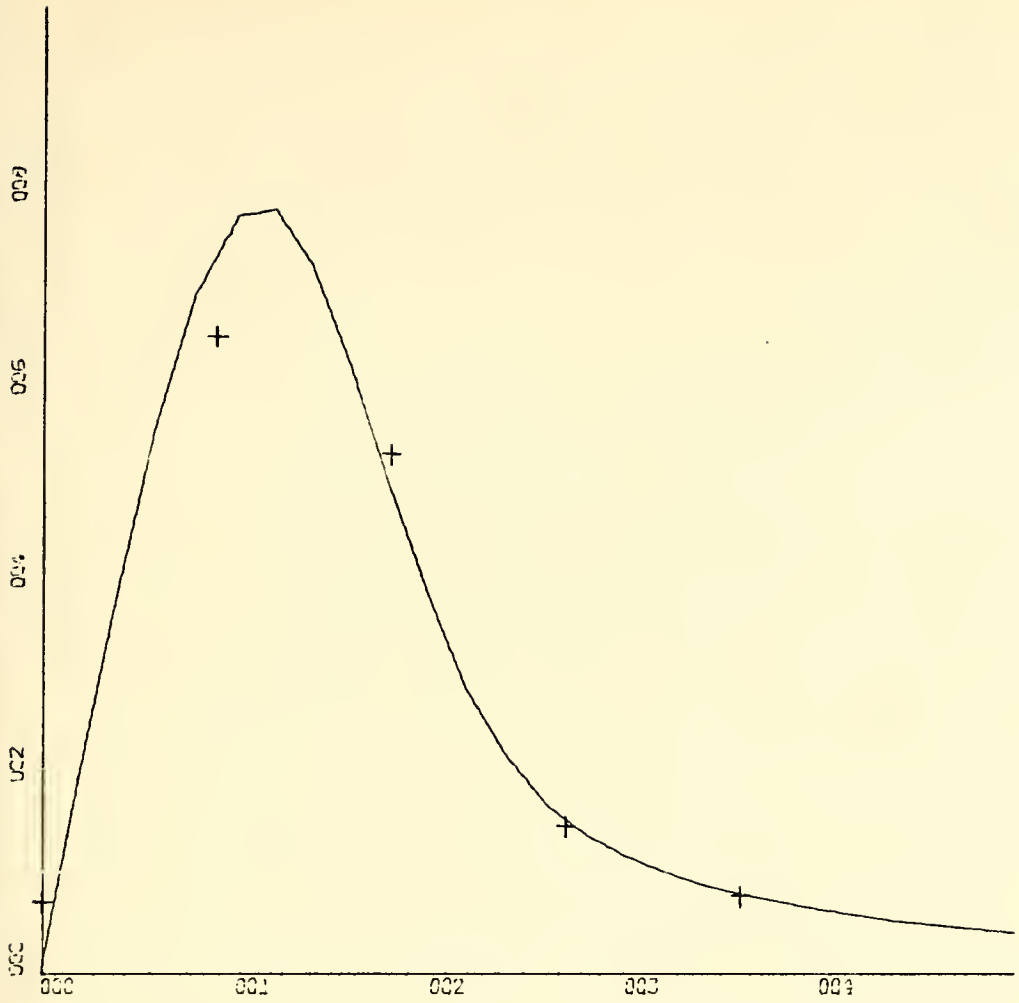
FIGURE 40



FOR DELTA T = 0.8 TFIN = 5.2

T	Q(N)
0.0	0.06667
0.8	0.63694
1.6	0.62939
2.4	0.23756
3.2	0.09620
4.0	0.06786

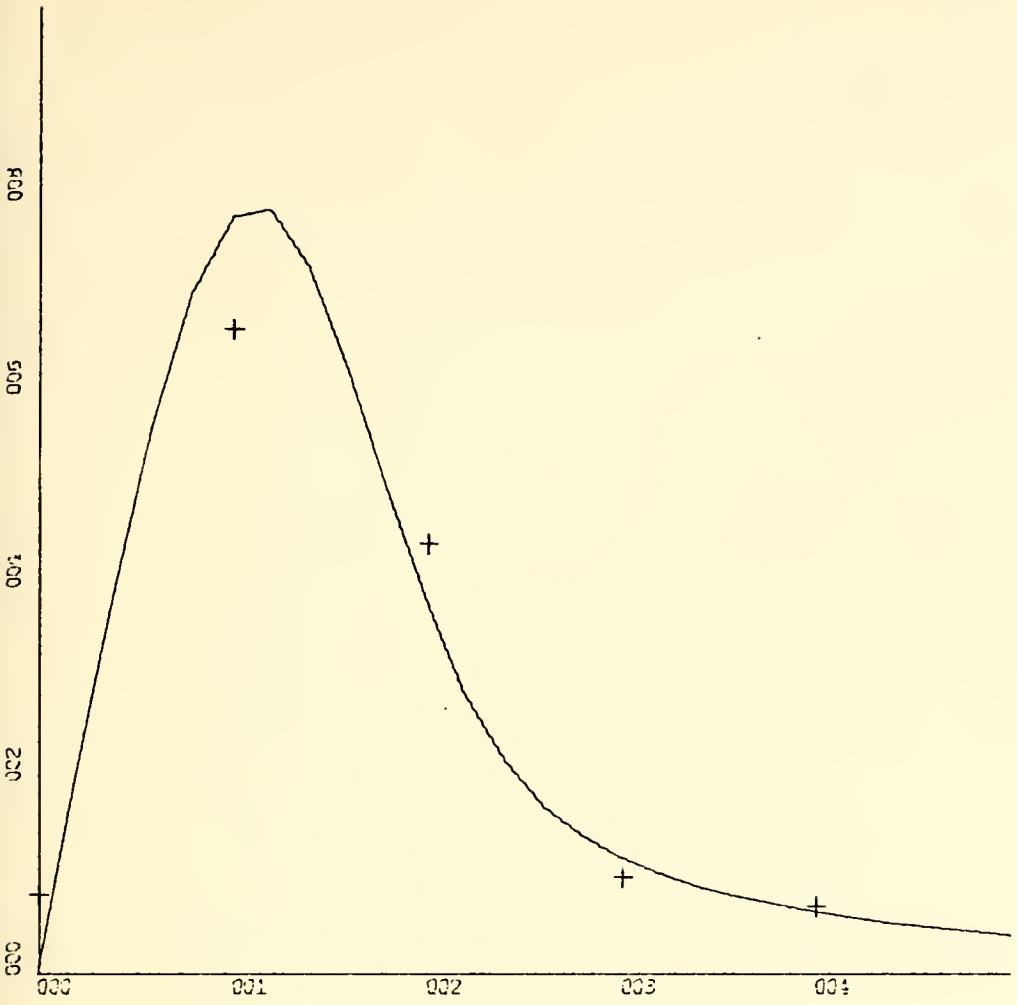
FIGURE 41



FOR DELTA T = 0.9 TFIN = 5.2

T	Q(N)
0.0	0.07500
0.9	0.65958
1.8	0.53668
2.7	0.15283
3.6	0.08072

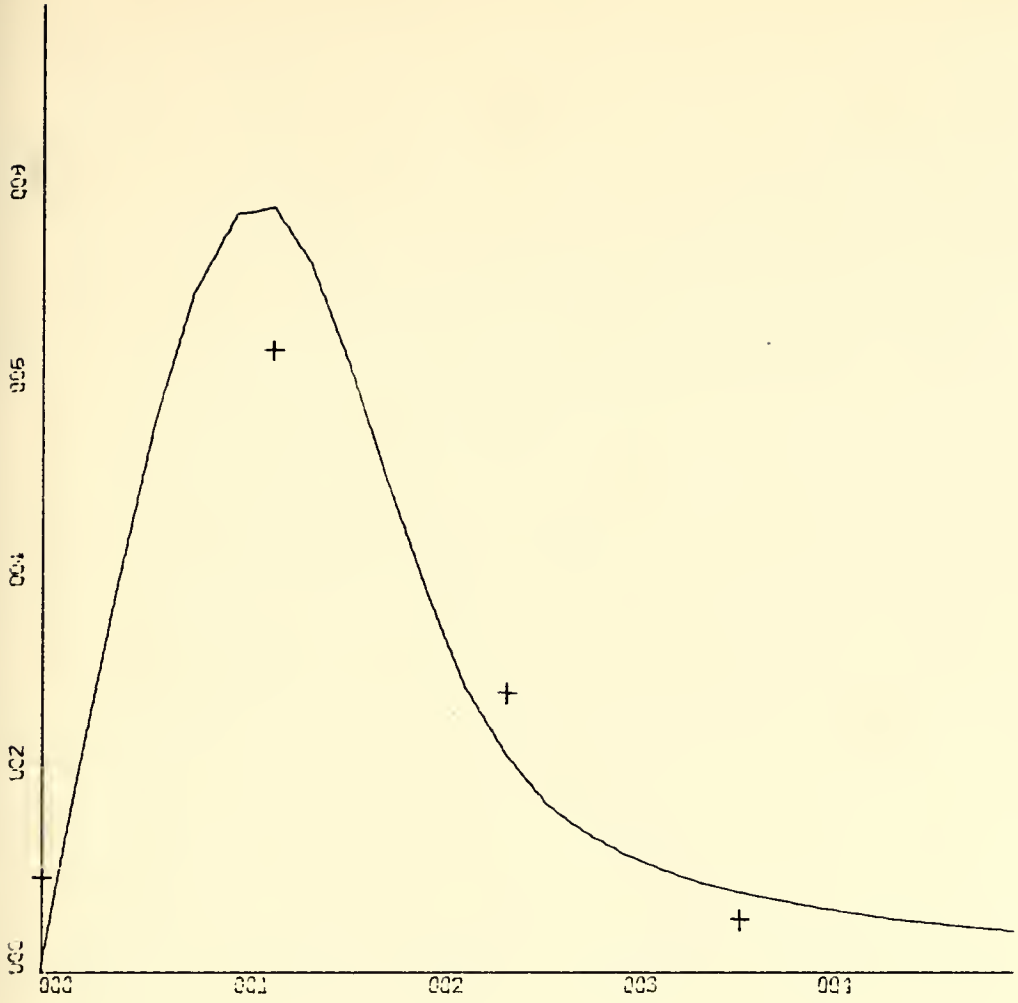
FIGURE 42



FOR DELTA T = 1.0 TFIN = 5.2

T	Q(N)
0.0	0.08333
1.0	0.66667
2.0	0.44444
3.0	0.10101
4.0	0.07183

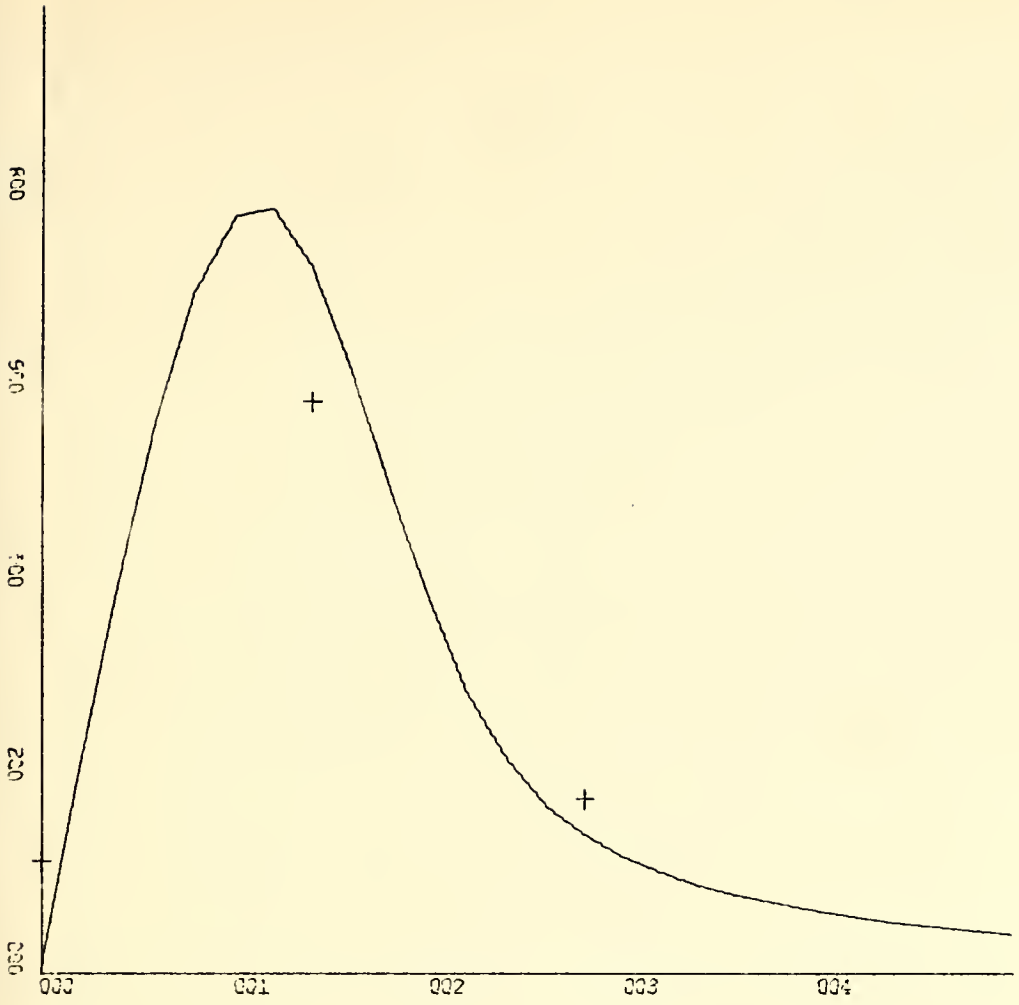
FIGURE 43



FOR DELTA T = 1.2 TFIN = 5.2

T	Q(N)
0.0	0.10000
1.2	0.64378
2.4	0.28895
3.6	0.05587

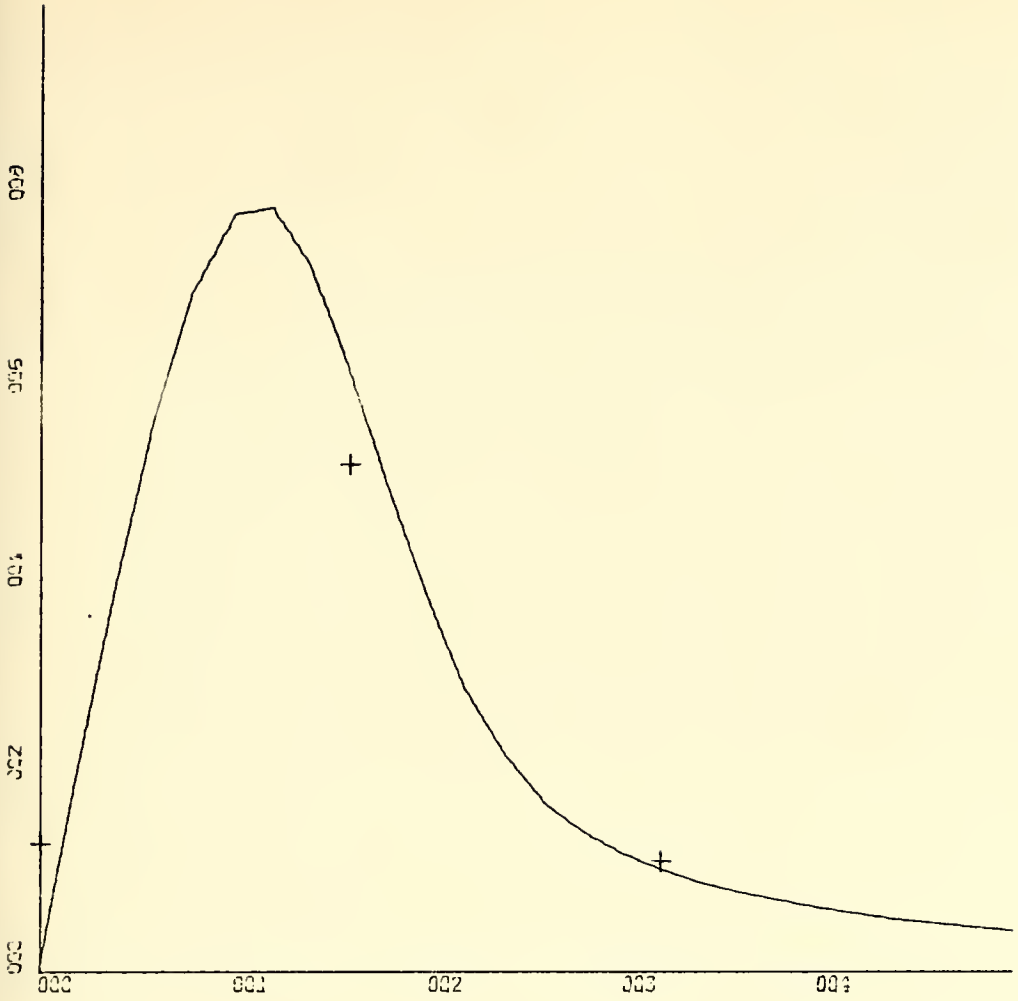
FIGURE 44



FOR DELTA T = 1.4 TFIN = 5.2

T	Q(N)
0.0	0.11667
1.4	0.59022
2.8	0.18194

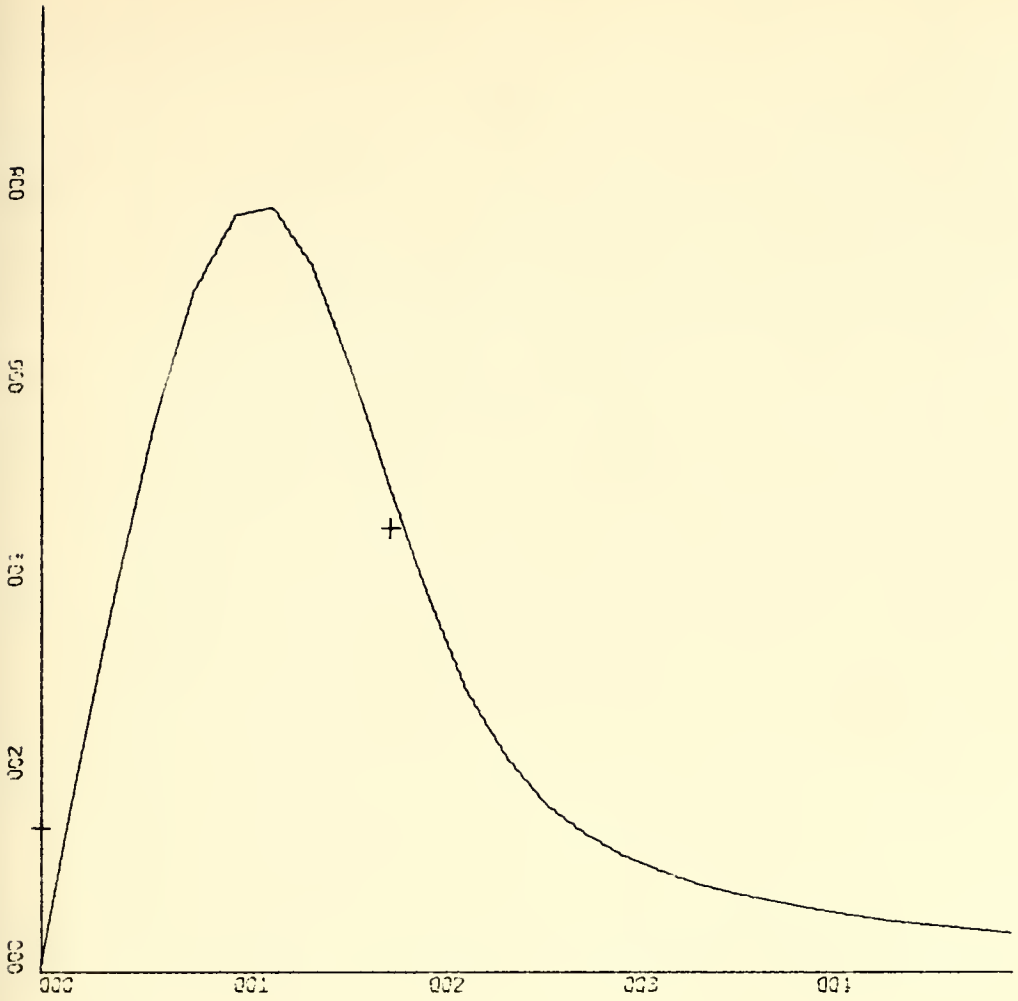
FIGURE 45



FOR DELTA T = 1.6 TFIN = 5.2

T	Q(N)
0.0	0.13333
1.6	0.52493
3.2	0.11422

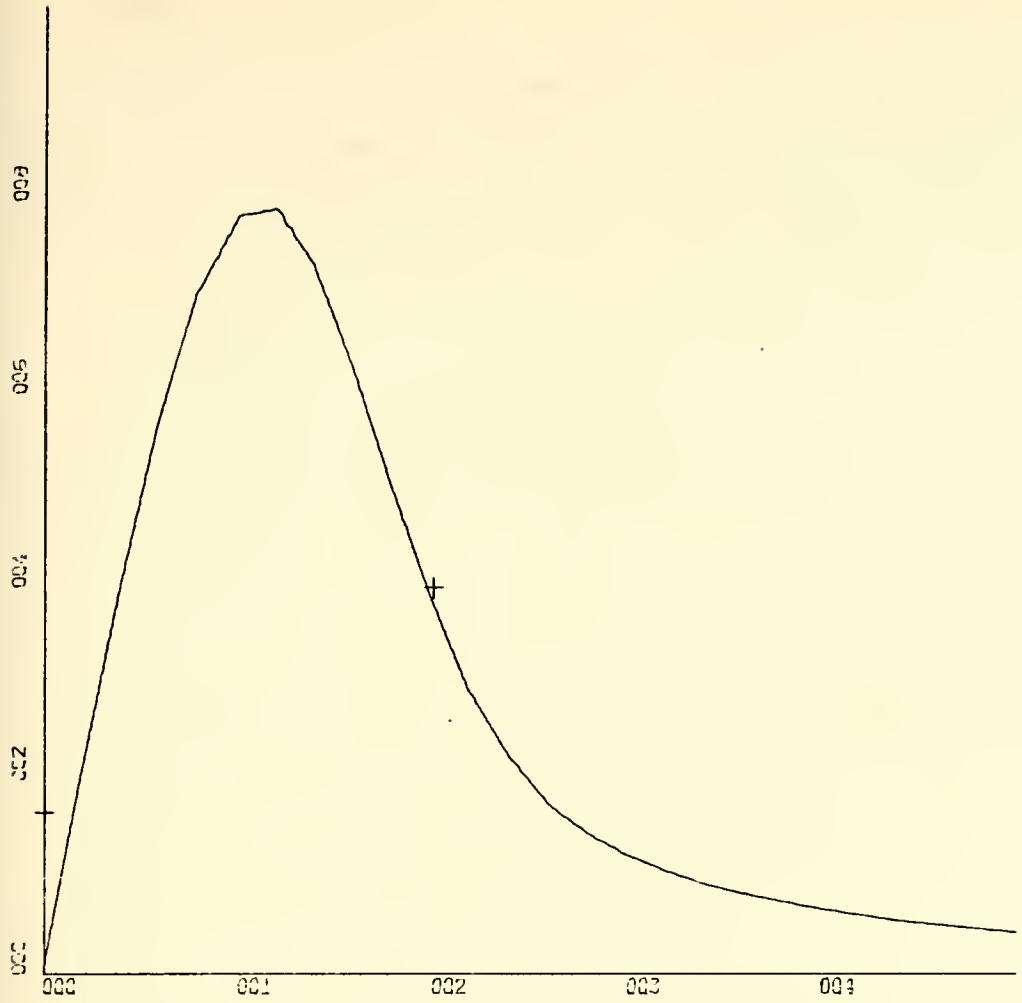
FIGURE 46



FOR DELTA T = 1.8 TFIN = 5.2

T	Q(N)
0.0	0.15000
1.8	0.45965

FIGURE 47



FOR DELTA T = 2.0 TFIN = 5.2

T	Q(N)
0.0	0.16667
2.0	0.40000

FIGURE 48

C. CASE 4 — $\dot{Y} + \sqrt{t}\,Y = t$

For this case, in the equation $\dot{Y} + P(t)Y = Q(t)$, $P(t) = \sqrt{t}$ and $Q(t) = t$, so following the procedure the equation is rewritten as:

$$\frac{dY}{dt} + PY = t \qquad Y_{(t=0)} = 0$$

where $P = \sqrt{t}$. The Laplace transform is:

$$sy + Py = 1/s^2$$

so

$$s^3y + Ps^2y = 1 \qquad y(s^3 + Ps^2) = 1$$

$$y = \frac{1}{s^3 + Ps^2} = \frac{s^{-3}}{1 + Ps^{-1}}$$

Substituting the z-forms of s^{-k} from Table I one gets:

$$y_A^*(z) = \frac{\frac{T^3}{2}\,\frac{z^{-1} + z^{-2}}{(1 - z^{-1})^3}}{1 + P\,\frac{T}{2}\,\frac{1 + z^{-1}}{1 - z^{-1}}}$$

Dividing by T and ordering in crescent powers of z^{-1} in numerator and denominator:

$$y_A^*(z) = \frac{T^2z^{-1} + T^2z^{-2}}{(2+Pt) - (6+PT)z^{-1} + (6-PT)z^{-2} + (PT-2)z^{-3}} \qquad (46)$$

Now, substituting for the time iteration wanted and replacing $P = \sqrt{t}$ for its value at the division step, the solution is obtained.

In the same way as before, plots and outputs follow for time iterations from 0.2 sec. to 2.0 sec. (Figs. 49 to 60).

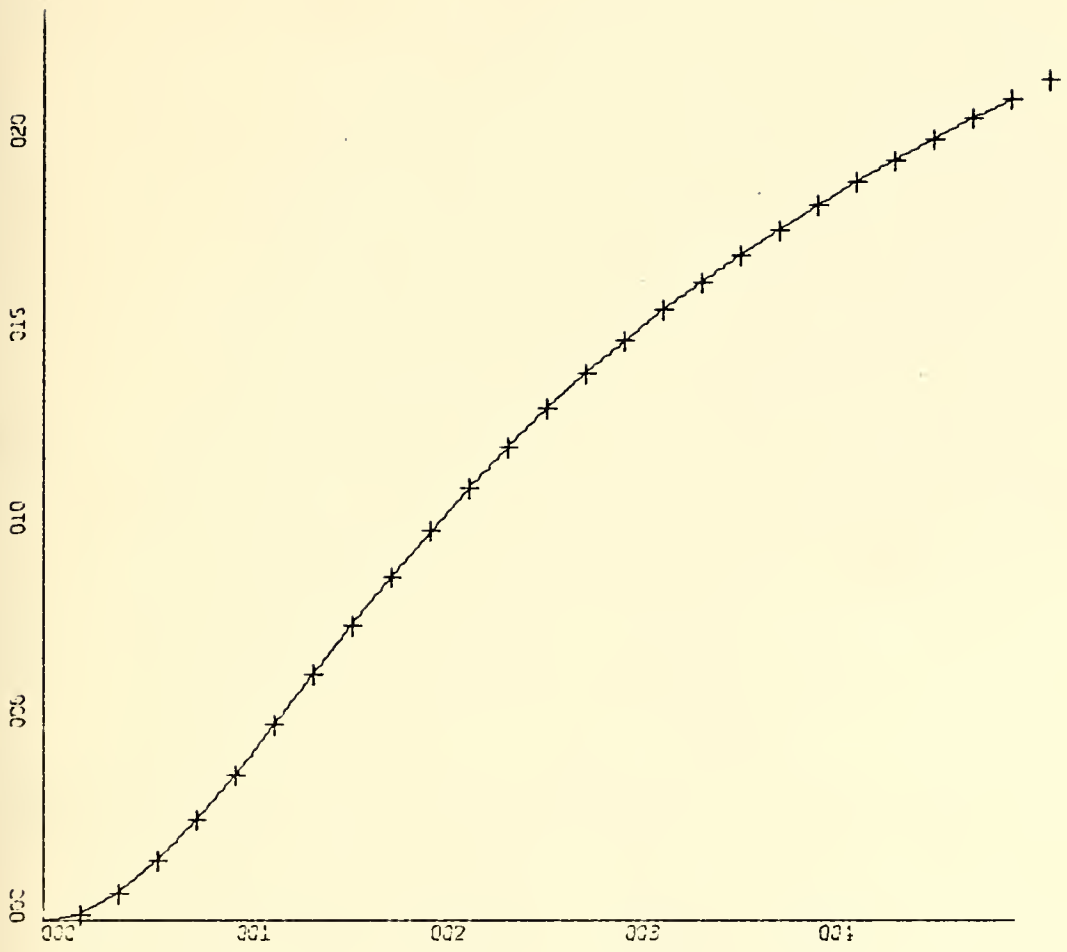
Algorithm for computer solution of this problem is shown in Program 4 (page 80).

FOR DELTA T = 0.2 TFIN = 5.2

T	Q(N)	PR
0.2	0.01914	0.0
0.4	0.07363	0.01949
0.6	0.15683	0.07449
0.8	0.26131	0.15817
1.0	0.37994	0.26299
1.2	0.50647	0.38181
1.4	0.63576	0.50837
1.6	0.76391	0.63756
1.8	0.88813	0.76551
2.0	1.00662	0.88948
2.2	1.11838	1.00770
2.4	1.22302	1.11920
2.6	1.32060	1.22360
2.8	1.41147	1.32100
3.0	1.49613	1.41170
3.2	1.57520	1.49620
3.4	1.64928	1.57520
3.6	1.71898	1.64930
3.8	1.78485	1.71890
4.0	1.84739	1.78480
4.2	1.90703	1.84730
4.4	1.96413	1.90700
4.6	2.01902	1.96410
4.8	2.07196	2.01900
5.0	2.12316	2.07200
5.2	2.17281	2.12320

```
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
C   Q(T)=RAMP   &   P(T)=SQRT(T)
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      INTEGER *4ITB(12)/12*0/
      REAL *4RTB(28)/28*0.0/
      DIMENSION A(100), Q(100), X(100), XX(100), PR(100)
      ITB(3) = 5
      ITB(4) = 5
      WRITE (6,8)
C
      DO 1 I=1,26
      READ (5,7) XX(I),PR(I)
      WRITE (6,7) XX(I),PR(I)
    1 CONTINUE
C
    2 READ (5,7,END=4) TD,TFIN
      WRITE (6,5) TD,TFIN
      WRITE (6,6)
      X(1) = 0.0
      M = TFIN/TD
      A(1) = TD**2
      A(2) = TD**2
      A(3) = 0.0
      A(4) = 0.0
C
      DO 3 N=1,M
      T = N*TD
      P = SQRT(T)
      F1 = 2.0+P*TD
      F2 = -(6.0+P*TD)
      F3 = 6.0-P*TD
      F4 = (P*TD)-2.0
C
      Q(N) = A(N)/F1
C
      A(N+1) = A(N+1)-(Q(N)*F2)
      A(N+2) = A(N+2)-(Q(N)*F3)
      A(N+3) = A(N+3)-(Q(N)*F4)
      A(N+4) = 0.0
      WRITE (6,7) T,Q(N)
      X(N) = T
    3 CONTINUE
C
      ITB(1) = 1
      ITB(2) = 0
      RTB(6) = TD
      ITB(12) = 1
      CALL DRAWP (26,XX,PR,ITB,RTB)
      ITB(1) = 3
      ITB(2) = 2
      CALL DRAWP (M,X,Q,ITB,RTB)
      GO TO 2
    4 STOP
C
    5 FORMAT ('1',' FOR DELTA T=',F6.2,4X,'TFIN=',F4.1)
    6 FORMAT ('   T=',7X,' Q(N)=')
    7 FORMAT (2F10.5)
    8 FORMAT ('  XX=',7X,'    PR=')
      END
```

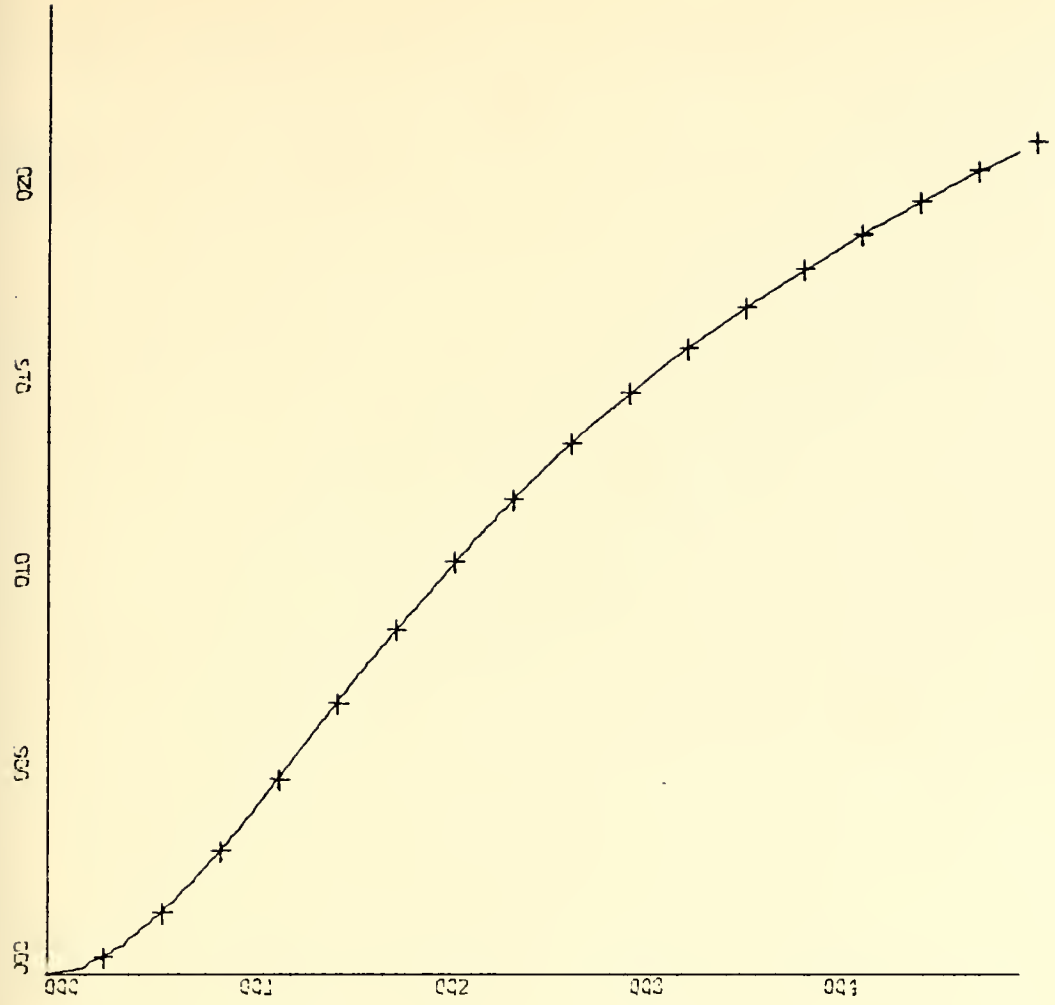
PROGRAM 4



x-scale = 1.0 units/inch
y-scale = 0.5 units/inch

FOR DELTA T = 0.2

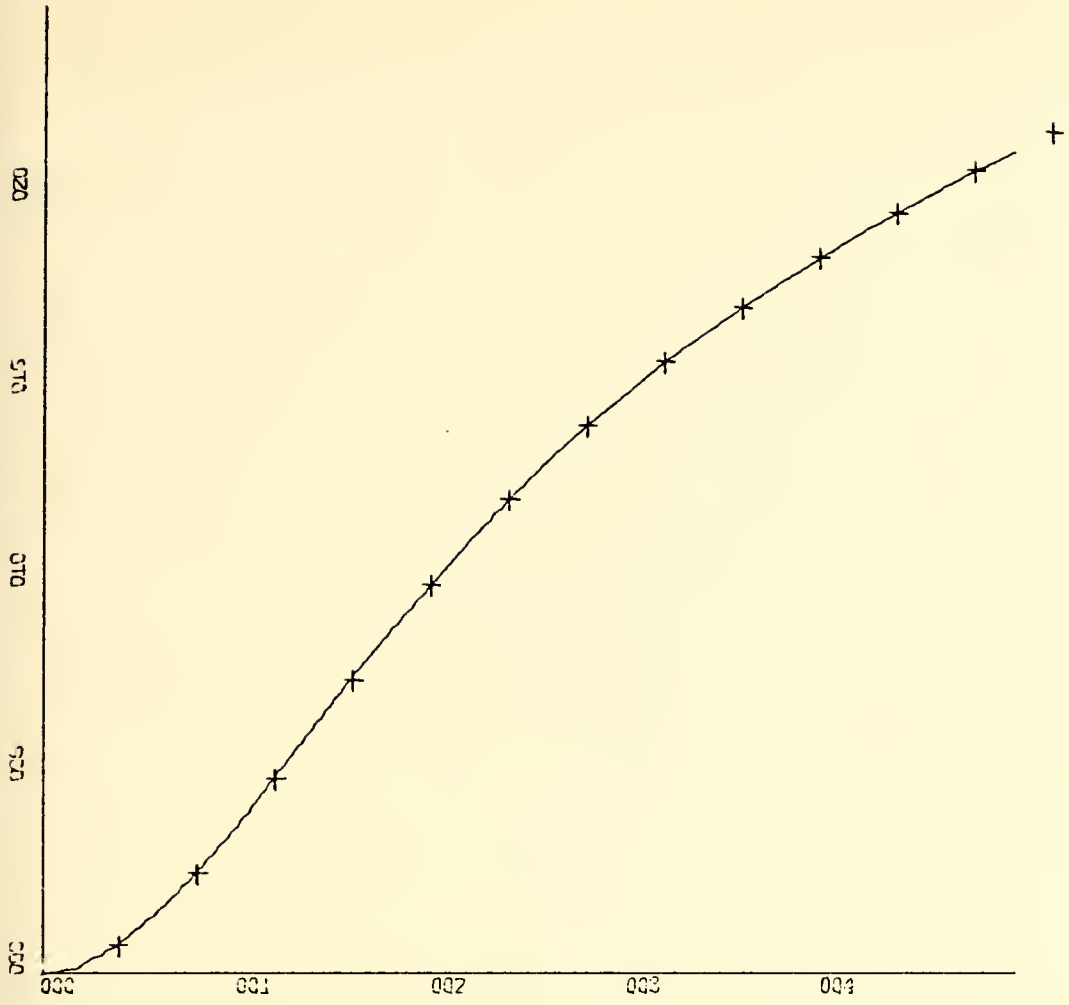
FIGURE 49



FOR DELTA T = 0.3 TFIN = 5.2

T	Q(N)
0.3	0.04158
0.6	0.15514
0.9	0.31700
1.2	0.50407
1.5	0.69801
1.8	0.88639
2.1	1.06213
2.4	1.22223
2.7	1.36641
3.0	1.49595
3.3	1.61280
3.6	1.71905
3.9	1.81664
4.2	1.90718
4.5	1.99200
4.8	2.07212
5.1	2.14834

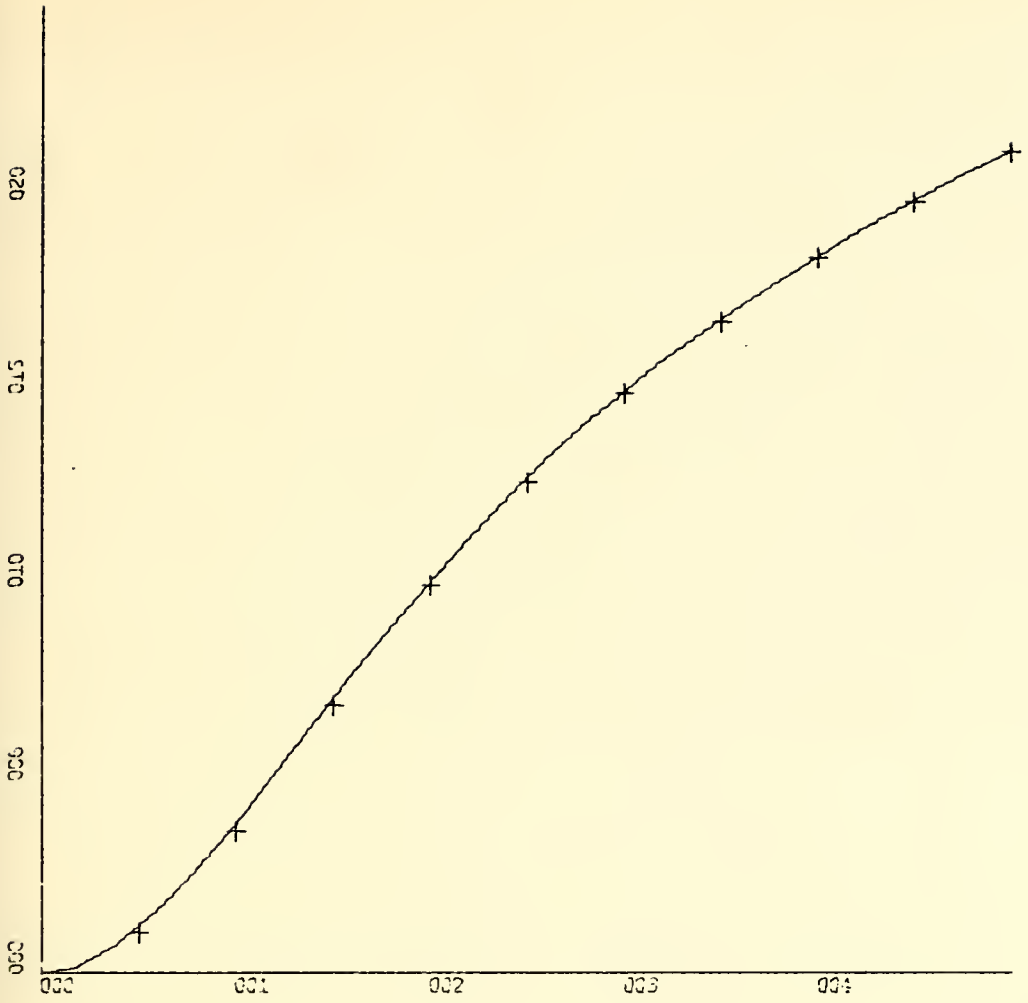
FIGURE 50



FOR DELTA T = 0.4 TFIN = 5.2

T	Q(N)
0.4	0.07102
0.8	0.25620
1.2	0.50068
1.6	0.75898
2.0	1.00322
2.4	1.22111
2.8	1.41066
3.2	1.57505
3.6	1.71916
4.0	1.84769
4.4	1.96446
4.8	2.07227
5.2	2.17312

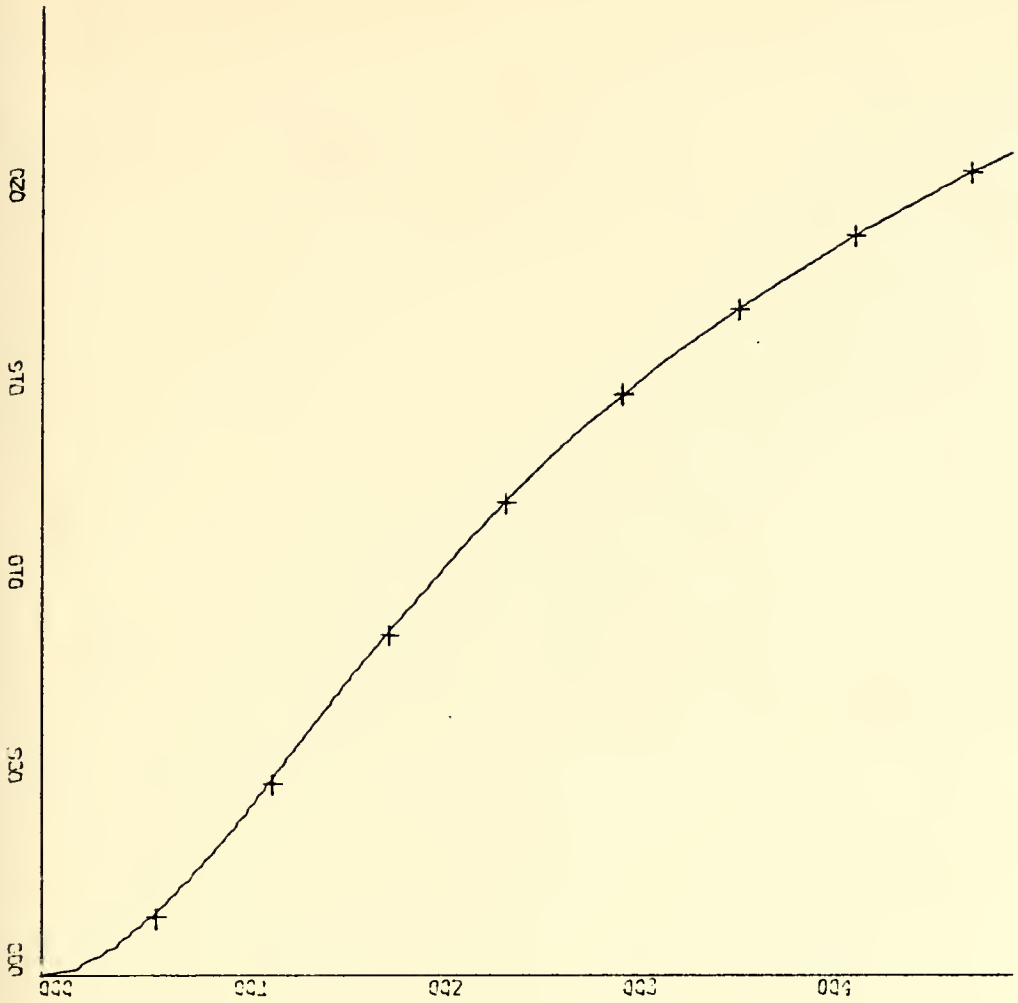
FIGURE 51



FOR DELTA T = 0.5 TFIN = 5.2

T	Q(N)
0.5	0.10622
1.0	0.36996
1.5	0.69092
2.0	1.00060
2.5	1.26987
3.0	1.49539
3.5	1.68484
4.0	1.84788
4.5	1.99233
5.0	2.12360

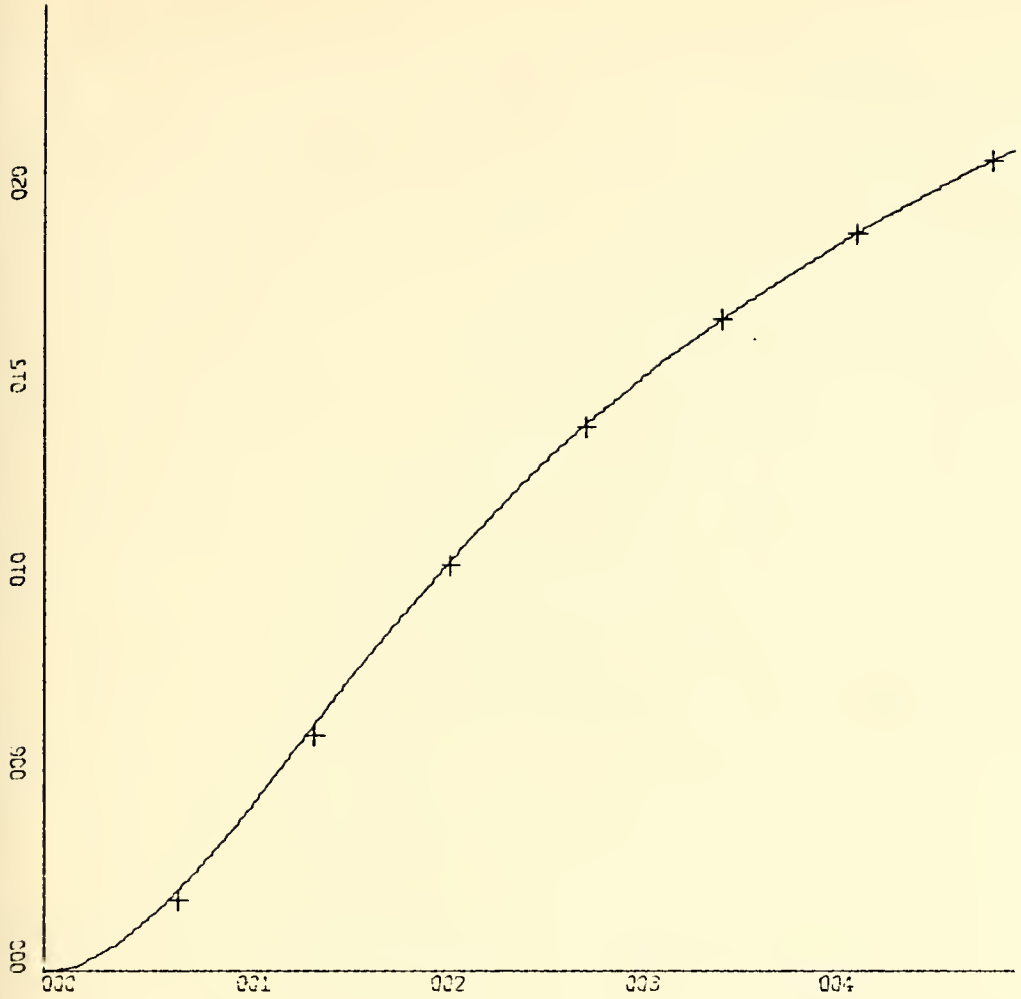
FIGURE 52



FOR DELTA T = 0.6 TFIN = 5.2

T	Q(N)
0.6	0.14606
1.2	0.49082
1.8	0.87667
2.4	1.21782
3.0	1.49500
3.6	1.71944
4.2	1.90777
4.8	2.07258

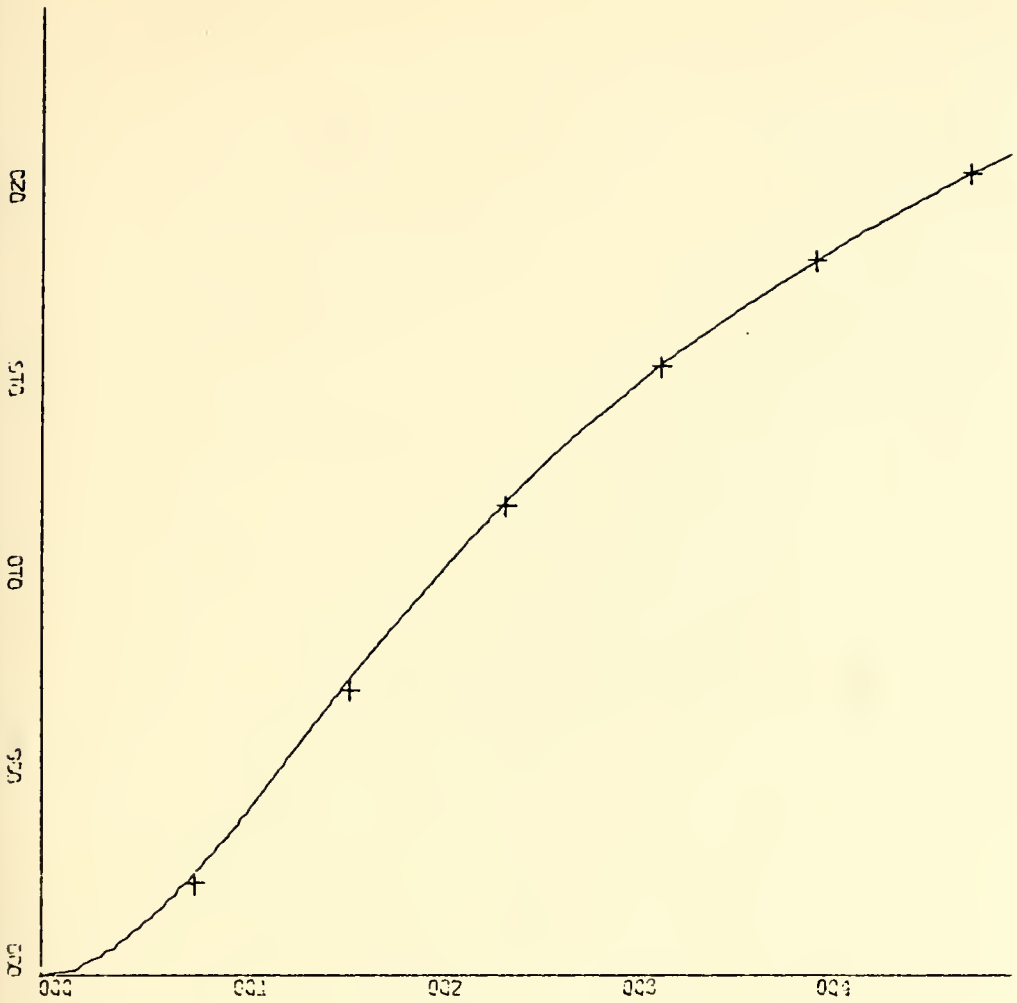
FIGURE 53



FOR DELTA T = 0.7 TFIN = 5.2

T	Q(N)
0.7	0.18951
1.4	0.61452
2.1	1.05164
2.8	1.40840
3.5	1.68514
4.2	1.90808
4.9	2.09855

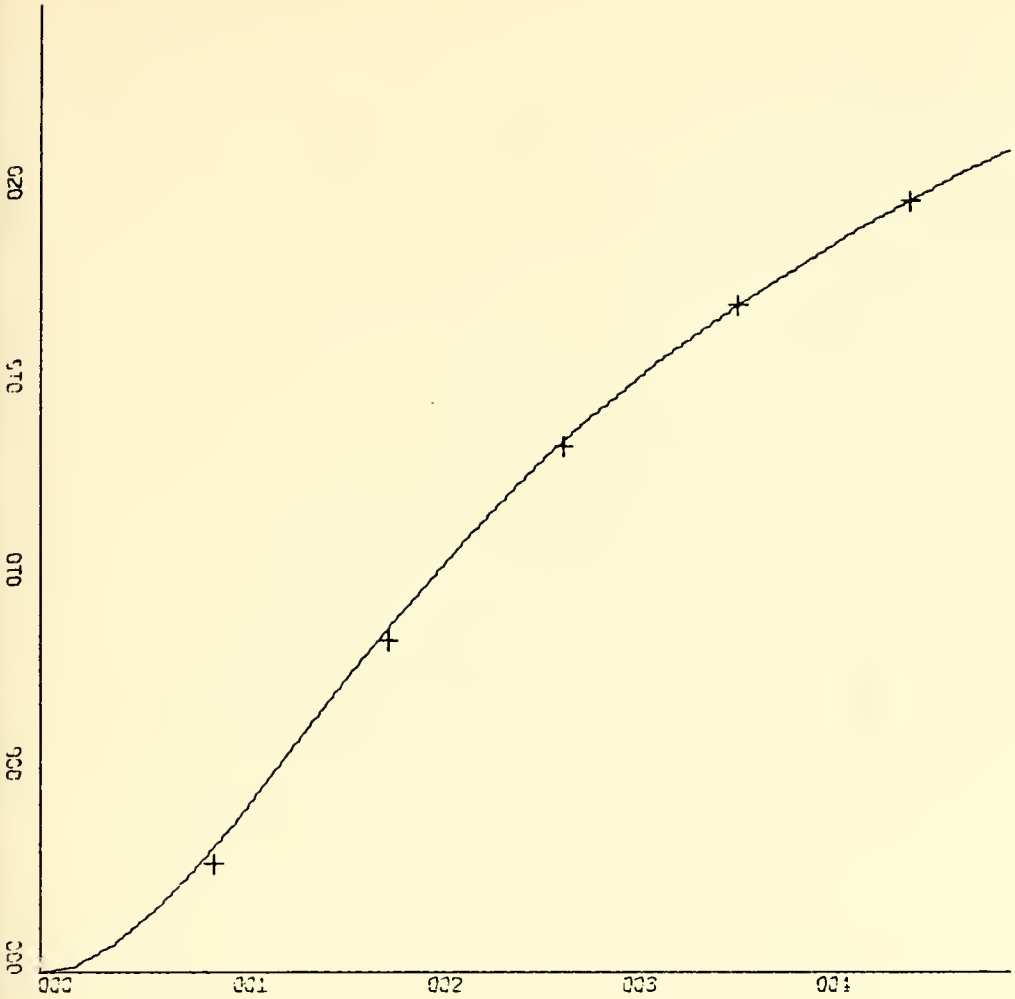
FIGURE 54



FOR DELTA T = 0.8 TFIN = 5.2

T	Q(N)
0.8	0.23568
1.6	0.73797
2.4	1.21295
3.2	1.57461
4.0	1.84884
4.8	2.07304

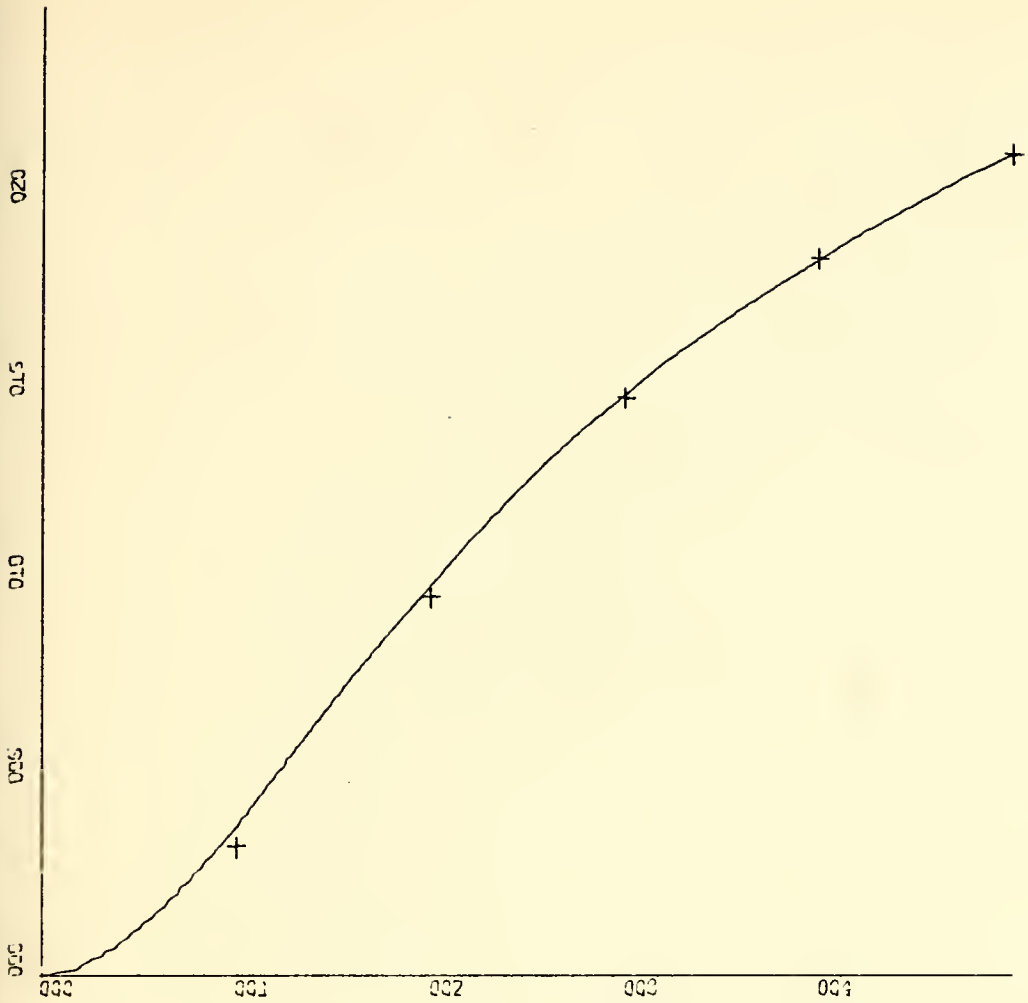
FIGURE 55



FOR DELTA T = 0.9 TFIN = 5.2

T	Q(N)
0.9	0.28383
1.8	0.85903
2.7	1.35987
3.6	1.72042
4.5	1.99350

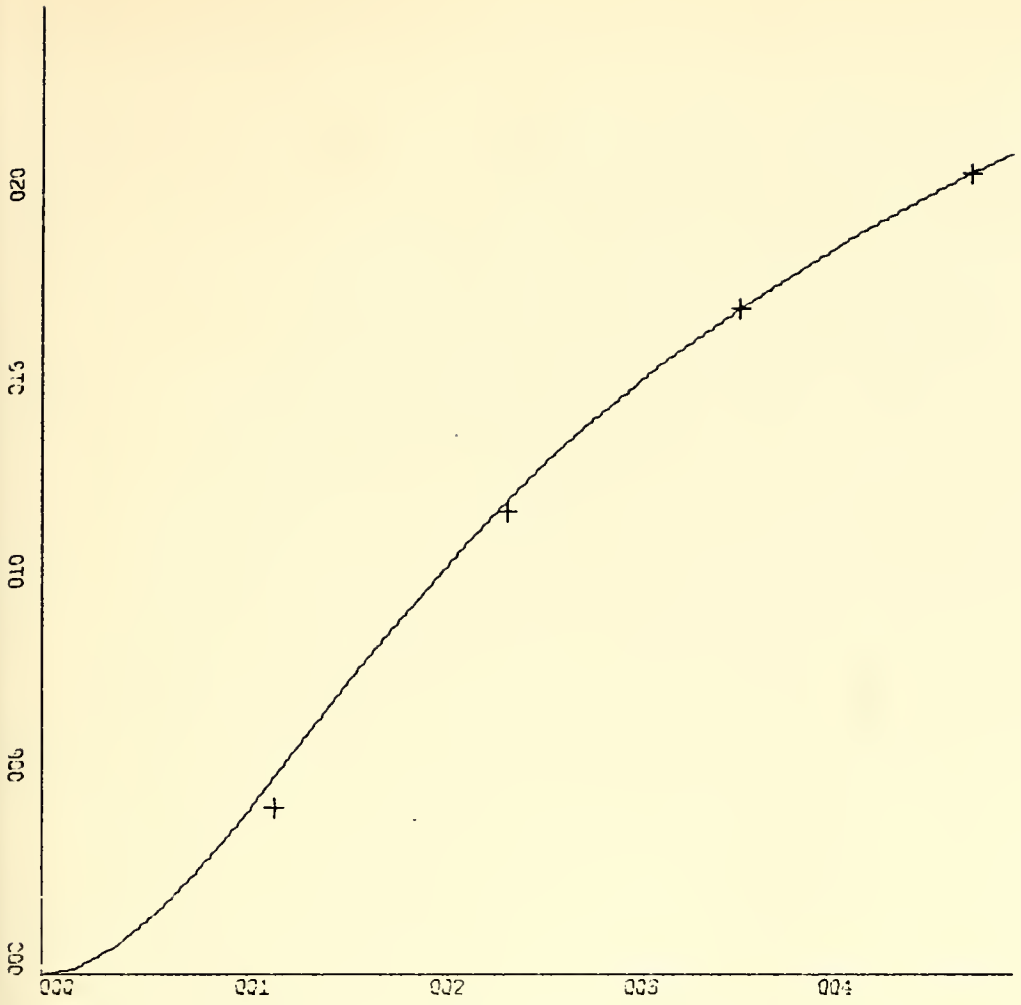
FIGURE 56



FOR DELTA T = 1.0 TFIN = 5.2

T	Q(N)
1.0	0.33333
2.0	0.97631
3.0	1.49299
4.0	1.85001
5.0	2.12461

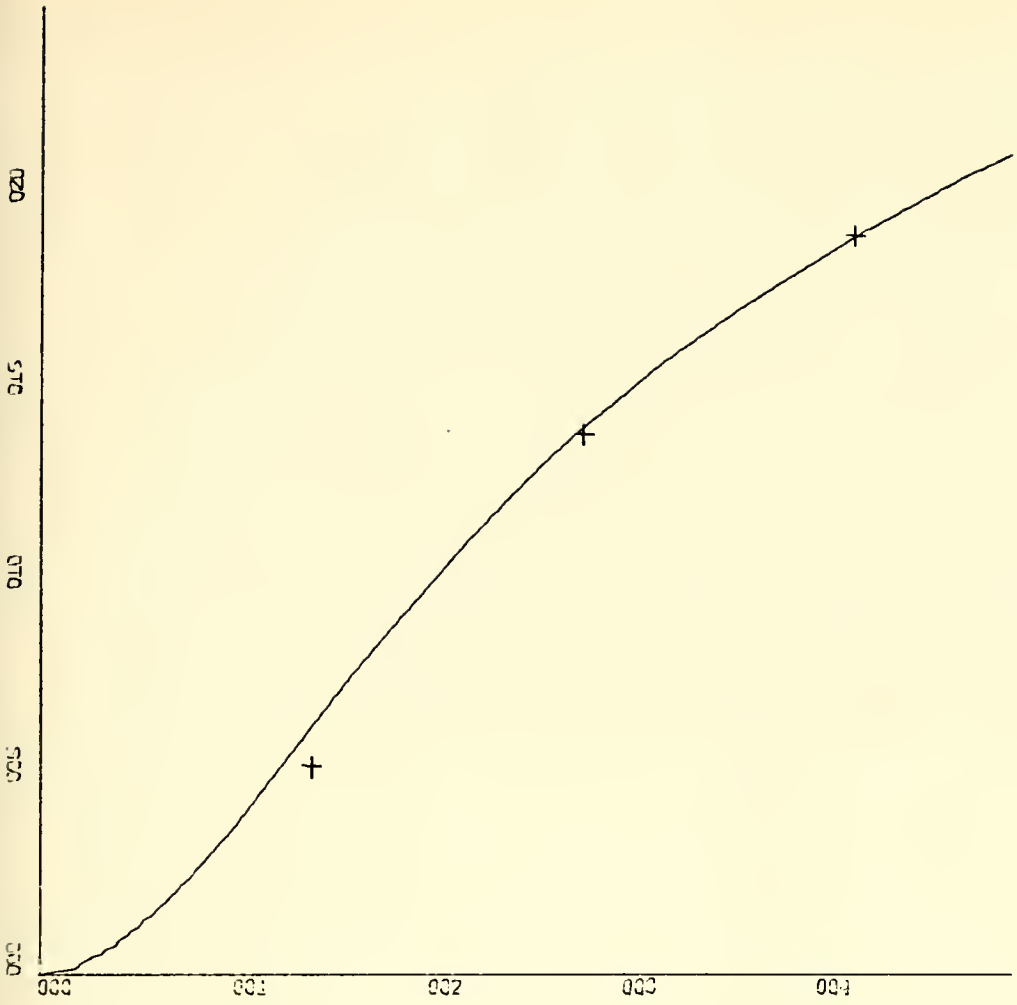
FIGURE 57



FOR DELTA T = 1.2 TFIN = 5.2

T	Q(N)
1.2	0.43445
2.4	1.19662
3.6	1.72293
4.8	2.07450

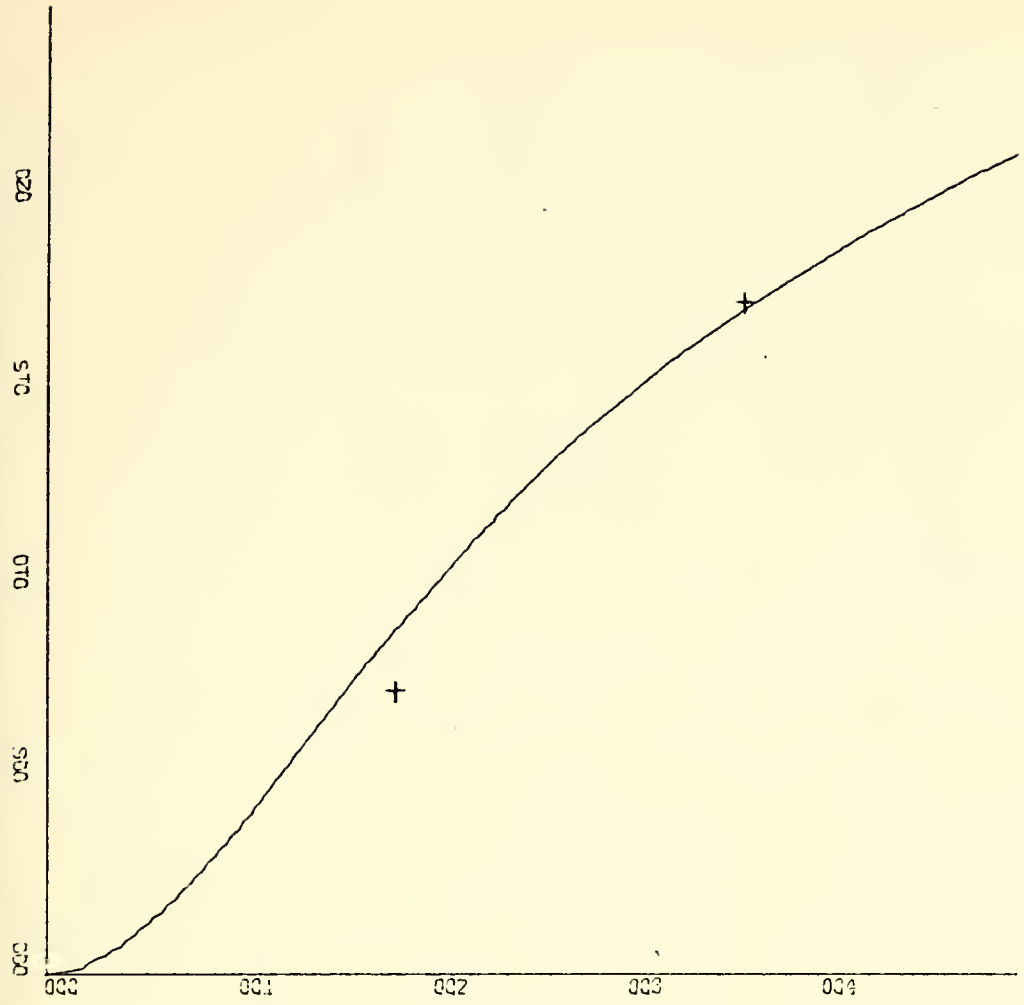
FIGURE 58



FOR DELTA T = 1.4 TFIN = 5.2

T	Q(N)
1.4	0.53603
2.8	1.39641
4.2	1.91440

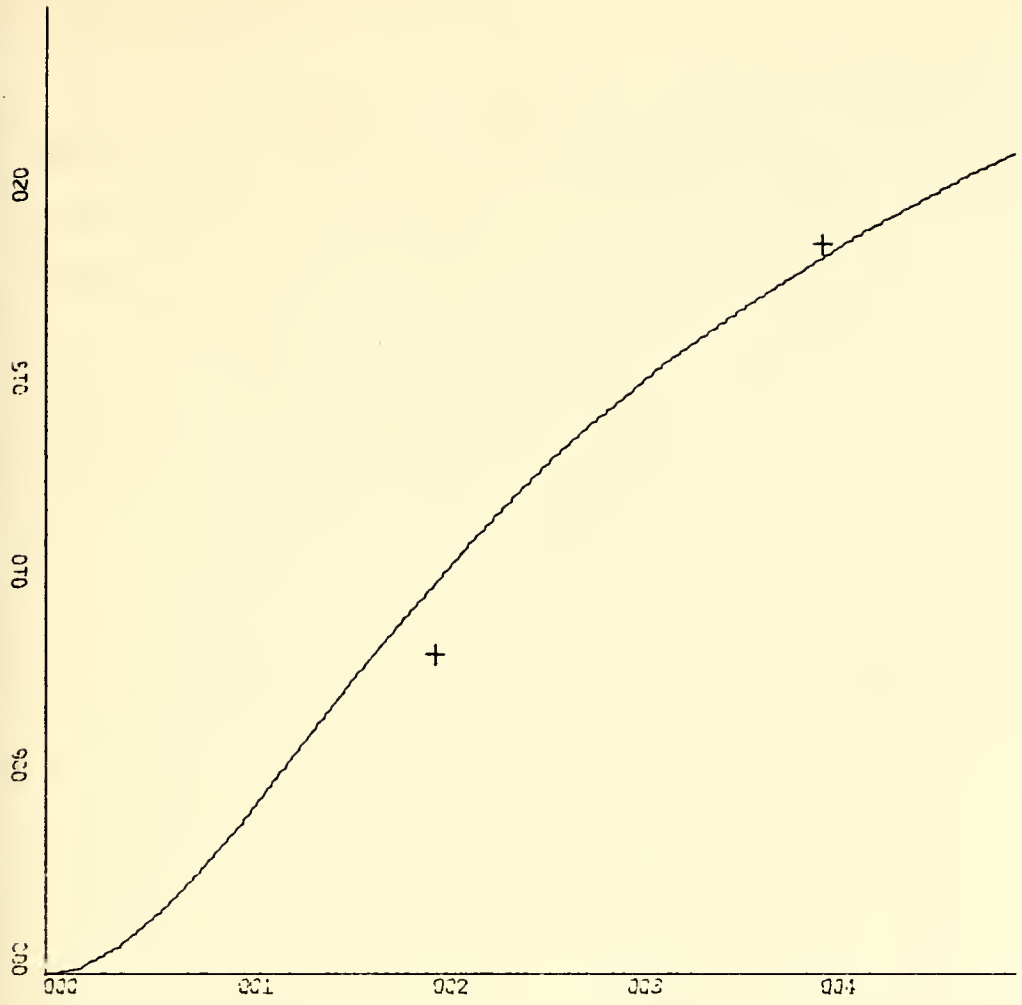
FIGURE 59



FOR DELTA T = 1.8 TFIN = 5.2

T	Q(N)
1.8	0.73387
3.6	1.73869

FIGURE 60



FOR DELTA T = 2. TFIN = 5.2

T	Q(N)
2.0	0.82843

FIGURE 61

1. Case 5 — $\dot{Y} + tY = t$

As case 2 follows case 1, as calculations are concerned, case 5 follows case 4 with the only difference that P(t) is now equal to t and this must be the value used to substitute in P before and during the division steps.

Graphs, plots and solution values for several time iterations follow (Figs. 62-75). Computer algorithm for solution of this problem is shown in Program 5 (page 95).

FOR DELTA T = 0.2 TFIN = 5.2

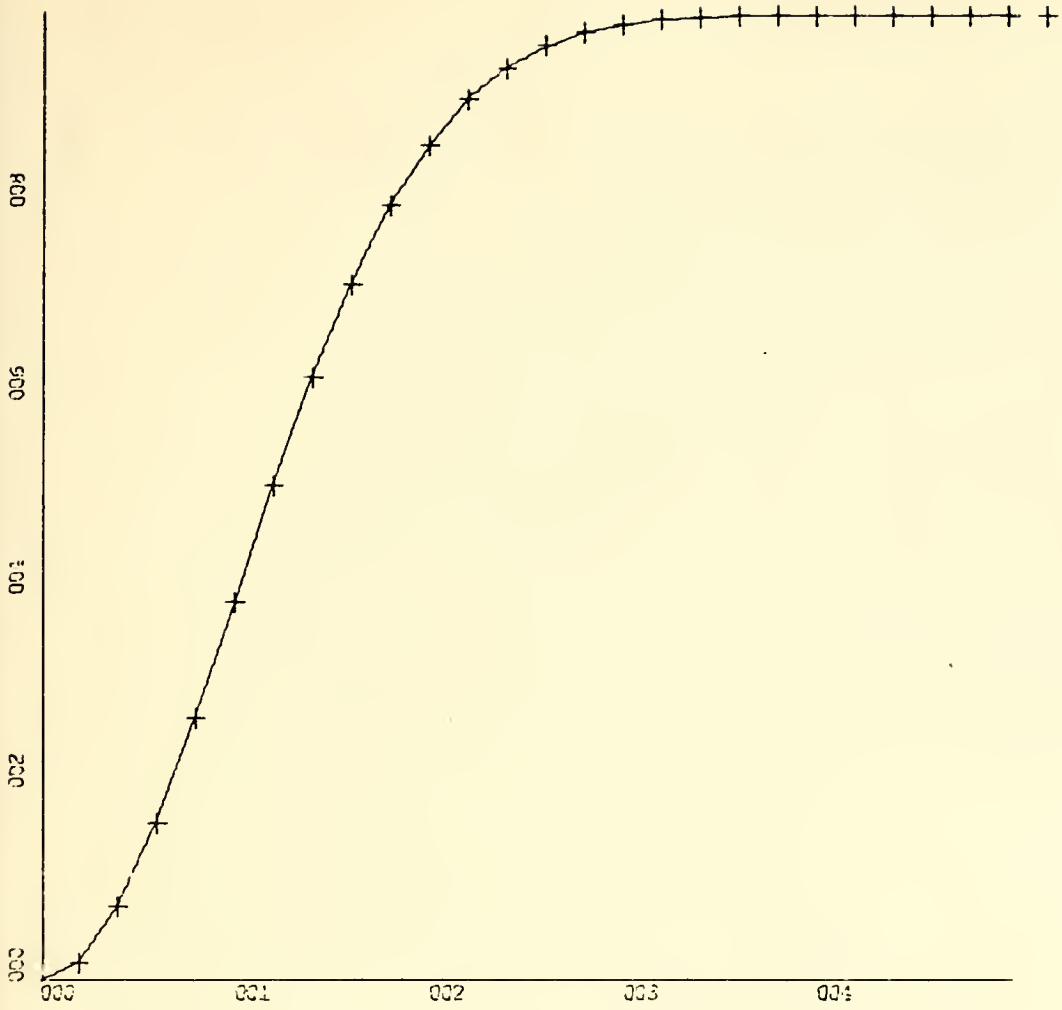
T	Q(N)	PR
0.0	0.0	0.0
0.2	0.01961	0.01980
0.4	0.07617	0.07688
0.6	0.16332	0.16473
0.8	0.27178	0.27385
1.0	0.39094	0.39347
1.2	0.51058	0.51324
1.4	0.62220	0.62468
1.6	0.71991	0.72186
1.8	0.80061	0.80209
2.0	0.86375	0.86466
2.2	0.91066	0.91107
2.4	0.94380	0.94386
2.6	0.96617	0.96595
2.8	0.98040	0.98016
3.0	0.98914	0.98889
3.2	0.99424	0.99402
3.4	0.99707	0.99691
3.6	0.99857	0.99847
3.8	0.99933	0.99927
4.0	0.99970	0.99966
4.2	0.99986	0.99985
4.4	0.99993	0.99994
4.6	0.99996	0.99997
4.8	0.99997	0.99999
5.0	0.99997	1.00000

Equation (46) is repeated here for convenience:

$$Y_A^*(z) = \frac{T^2 z^{-1} + T^2 z^{-2}}{(2+PT) - (6+PT)z^{-1} + (6-PT)z^{-2} + (PT-2)z^{-3}}$$

```
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC3C
C
C   Q(T)=RAMP   &   P(T)=T
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      INTEGER *4ITB(12)/12*0/
      REAL *4RTB(28)/28*0.0/
      DIMENSION A(100), Q(100), X(100), XX(100), PR(100)
      ITB(3) = 5
      ITB(4) = 5
      WRITE (6,8)
C
      DO 1 I=1,26
      READ (5,7) XX(I),PR(I)
      WRITE (6,7) XX(I),PR(I)
    1 CONTINUE
C
    2 READ (5,7,END=4) TD,TFIN
      WRITE (6,5) TD,TFIN
      WRITE (6,6)
      X(1) = 0.0
      M = TFIN/TD
      A(1) = TD**2
      A(2) = TD**2
      A(3) = 0.0
      A(4) = 0.0
C
      DO 3 N=1,M
      T = N*TD
      P = T
      F1 = 2.0+P*TD
      F2 = -(6.0+P*TD)
      F3 = 6.0-P*TD
      F4 = (P*TD)-2.0
C
      Q(N) = A(N)/F1
C
      A(N+1) = A(N+1)-(Q(N)*F2)
      A(N+2) = A(N+2)-(Q(N)*F3)
      A(N+3) = A(N+3)-(Q(N)*F4)
      A(N+4) = 0.0
      WRITE (6,7) T,Q(N)
      X(N) = T
    3 CONTINUE
C
      ITB(1) = 1
      ITB(2) = 0
      RTB(6) = TD
      ITB(12) = 1
      CALL DRAWP (26,XX,PR,ITB,RTB)
      ITB(1) = 3
      ITB(2) = 2
      CALL DRAWP (M,X,Q,ITB,RTB)
      GO TO 2
    4 STOP
C
    5 FORMAT ('1',' FOR DELTA T=',F6.2,4X,'TFIN=',F4.1)
    6 FORMAT ('  T=',7X,' Q(N)=')
    7 FORMAT (2F10.5)
    8 FORMAT ('  XX=',7X,'    PR=')
      END
```

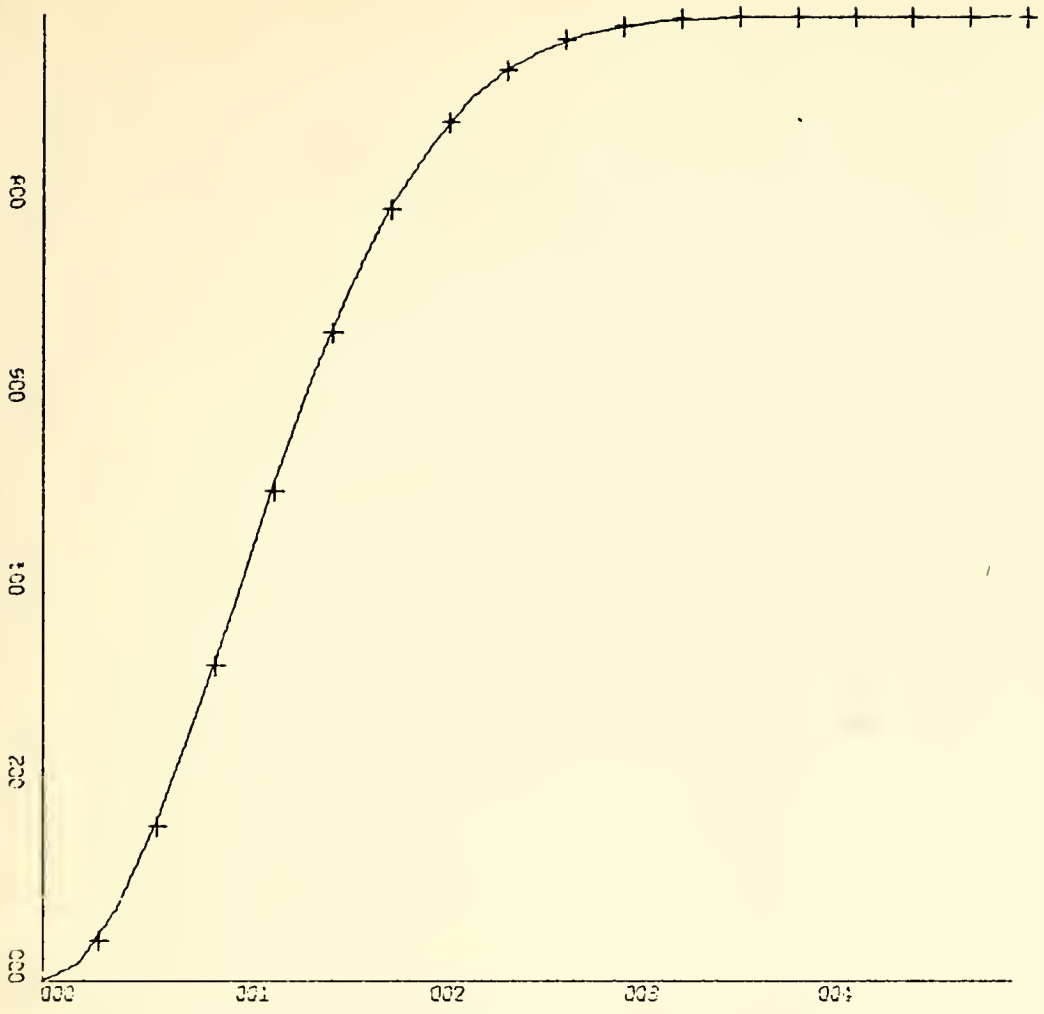
PROGRAM 5



x-scale = 1.0 units/inch
y-scale = 0.2 units/inch

DELTA T = 0.2

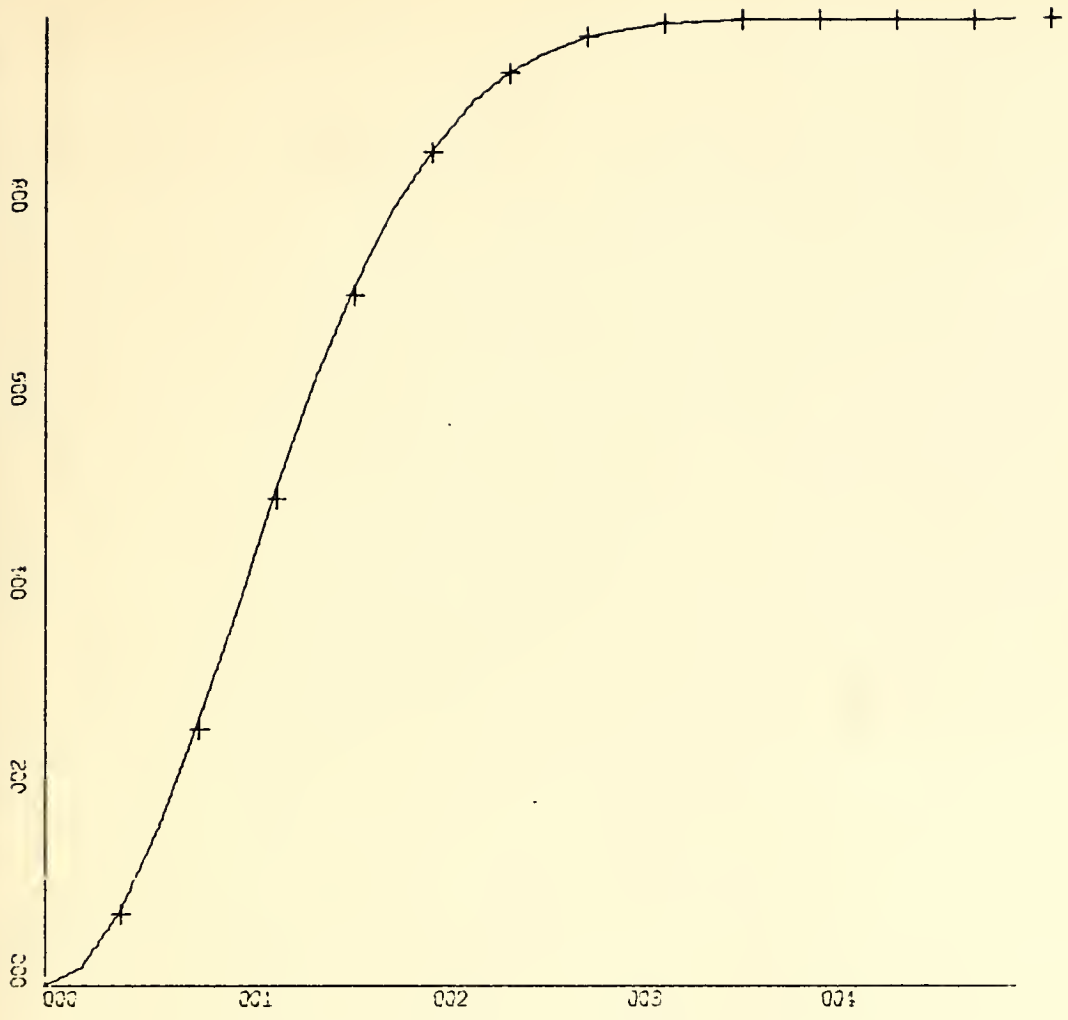
FIGURE 62



FOR DELTA T = 0.3 TFIN = 5.2

T	Q(N)
0.3	0.04306
0.6	0.16158
0.9	0.32779
1.2	0.50723
1.5	0.67015
1.8	0.79871
2.1	0.88826
2.4	0.94372
2.7	0.97436
3.0	0.98948
3.3	0.99613
3.6	0.99873
3.9	0.99963
4.2	0.99990
4.5	0.99997
4.8	0.99999
5.1	0.99999

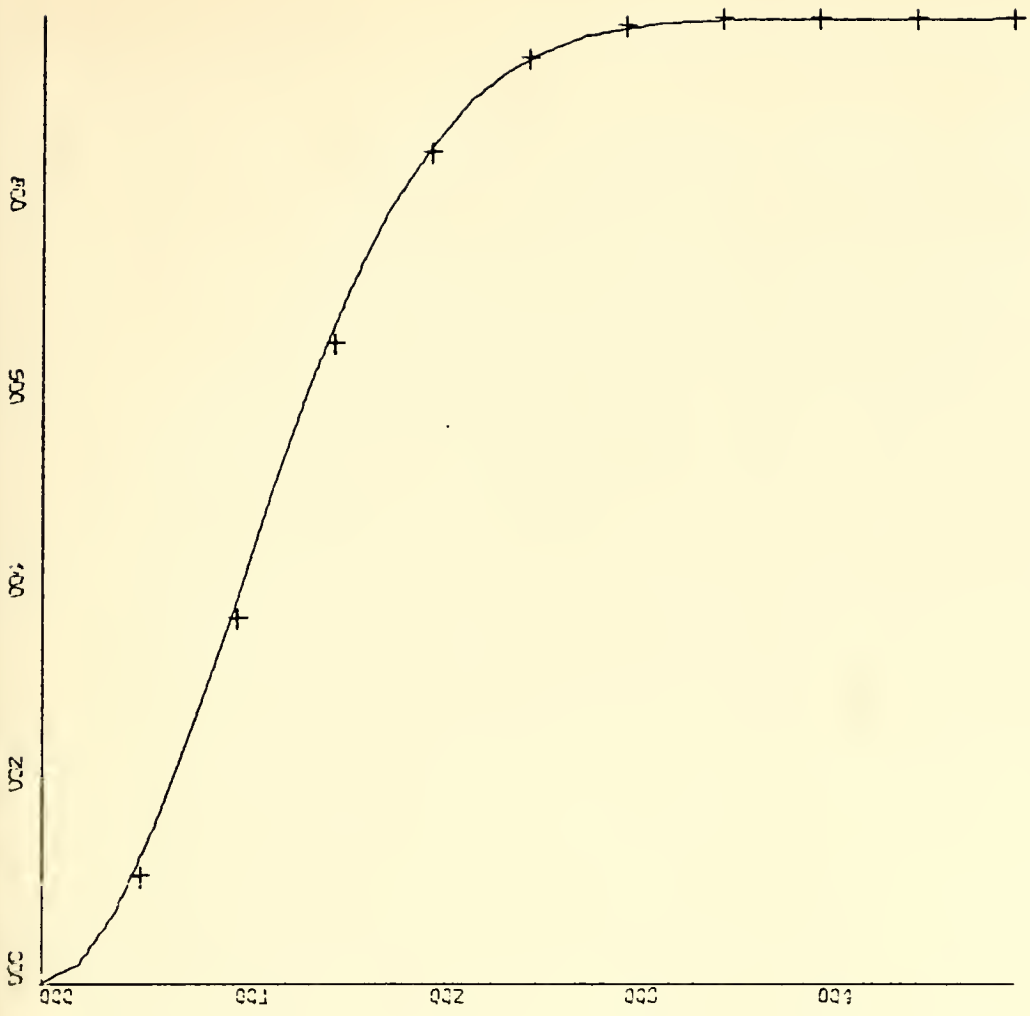
FIGURE 63



FOR DELTA T = 0.4 TFIN = 5.2

T	Q(N)
0.4	0.07407
0.8	0.26564
1.2	0.50253
1.6	0.71358
2.0	0.86088
2.4	0.94360
2.8	0.98120
3.2	0.99496
3.6	0.99895
4.0	0.99984
4.4	0.99998
4.8	1.00000
5.2	1.00000

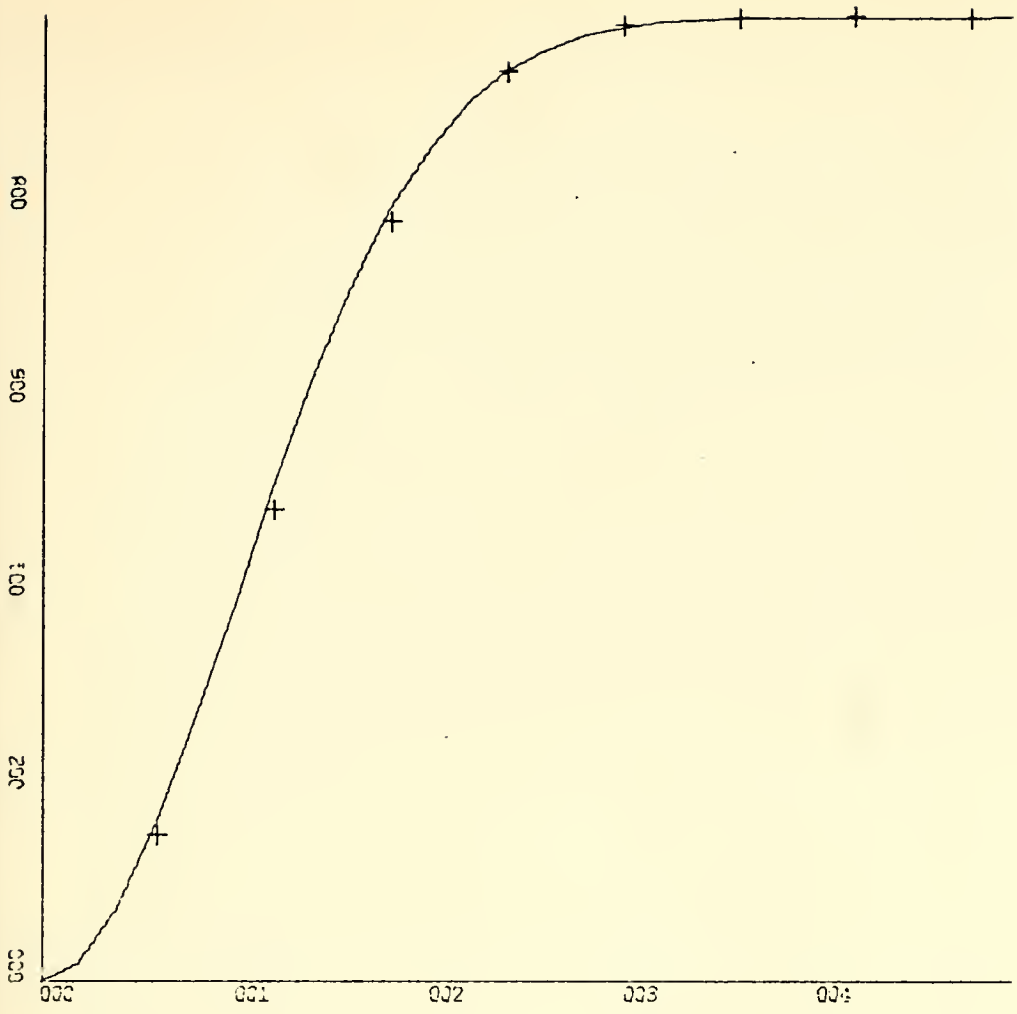
FIGURE 64



FOR DELTA T = 0.5 TFIN = 5.2

T	Q(N)
0.5	0.11111
1.0	0.37776
1.5	0.66061
2.0	0.85859
2.5	0.95649
3.0	0.99067
3.5	0.99876
4.0	0.99992
4.5	1.00000
5.0	1.00000

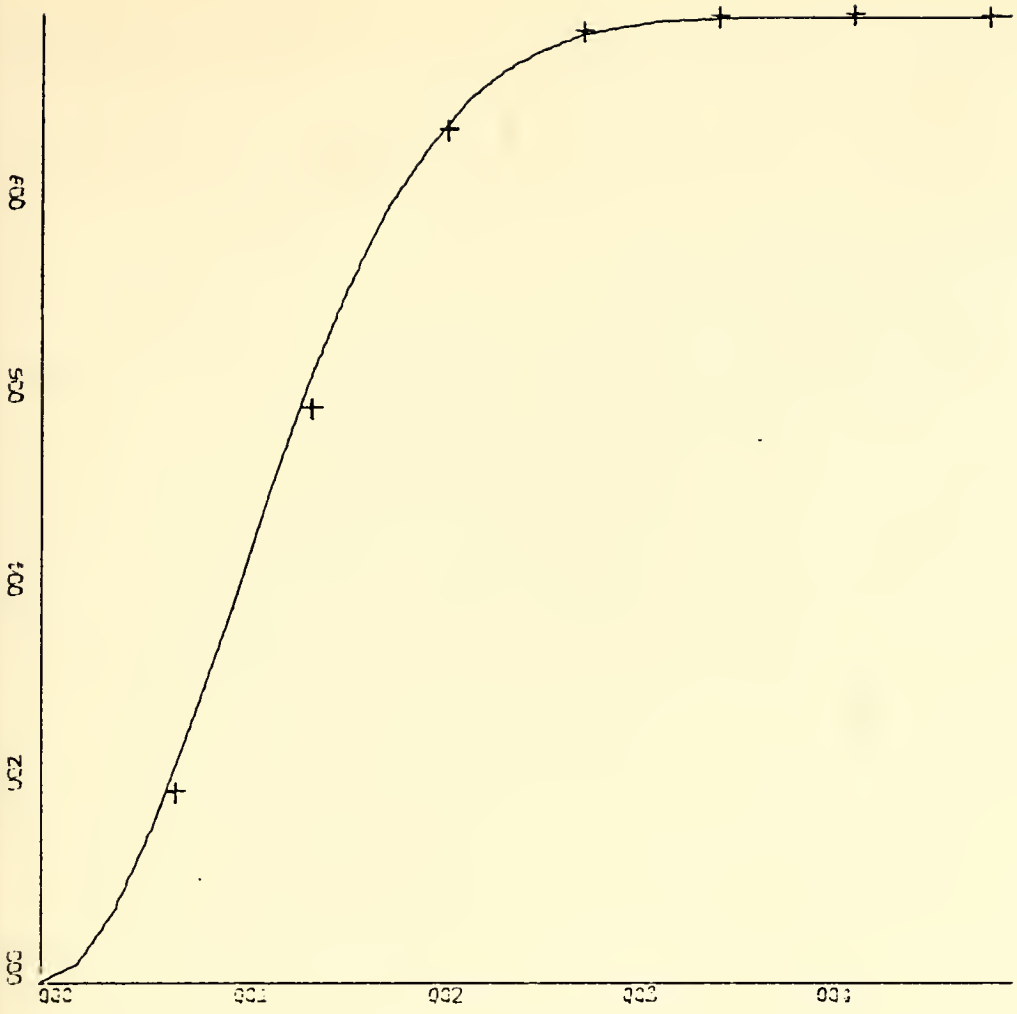
FIGURE 65



FOR DELTA T = 0.6 TFIN = 5.2

T	Q(N)
0.6	0.15254
1.2	0.48903
1.8	0.78765
2.4	0.94321
3.0	0.99163
3.6	0.99960
4.2	1.00001
4.8	1.00000

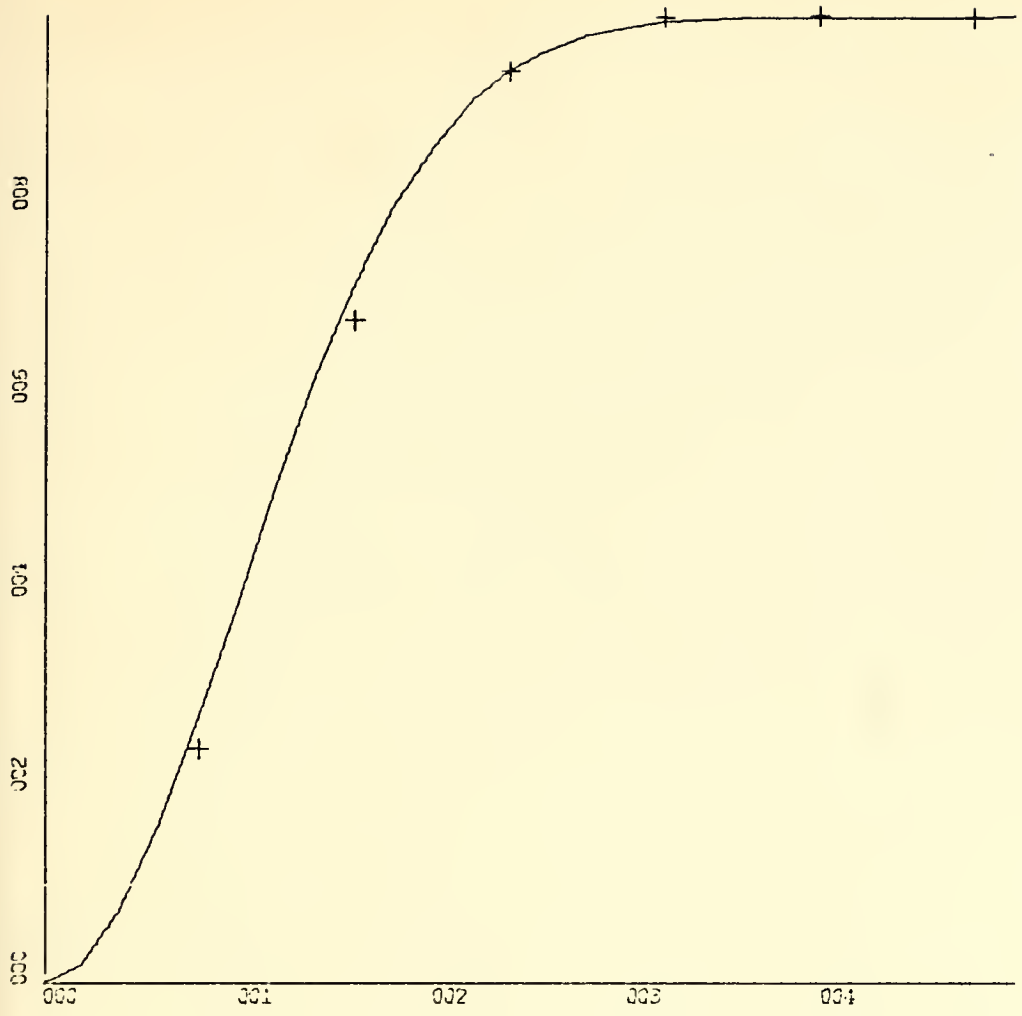
FIGURE 66



FOR DELTA T = 0.7 TFIN = 5.2

T	Q(N)
0.7	0.19679
1.4	0.59300
2.1	0.88036
2.8	0.98399
3.5	0.99986
4.2	1.00001
4.9	1.00000

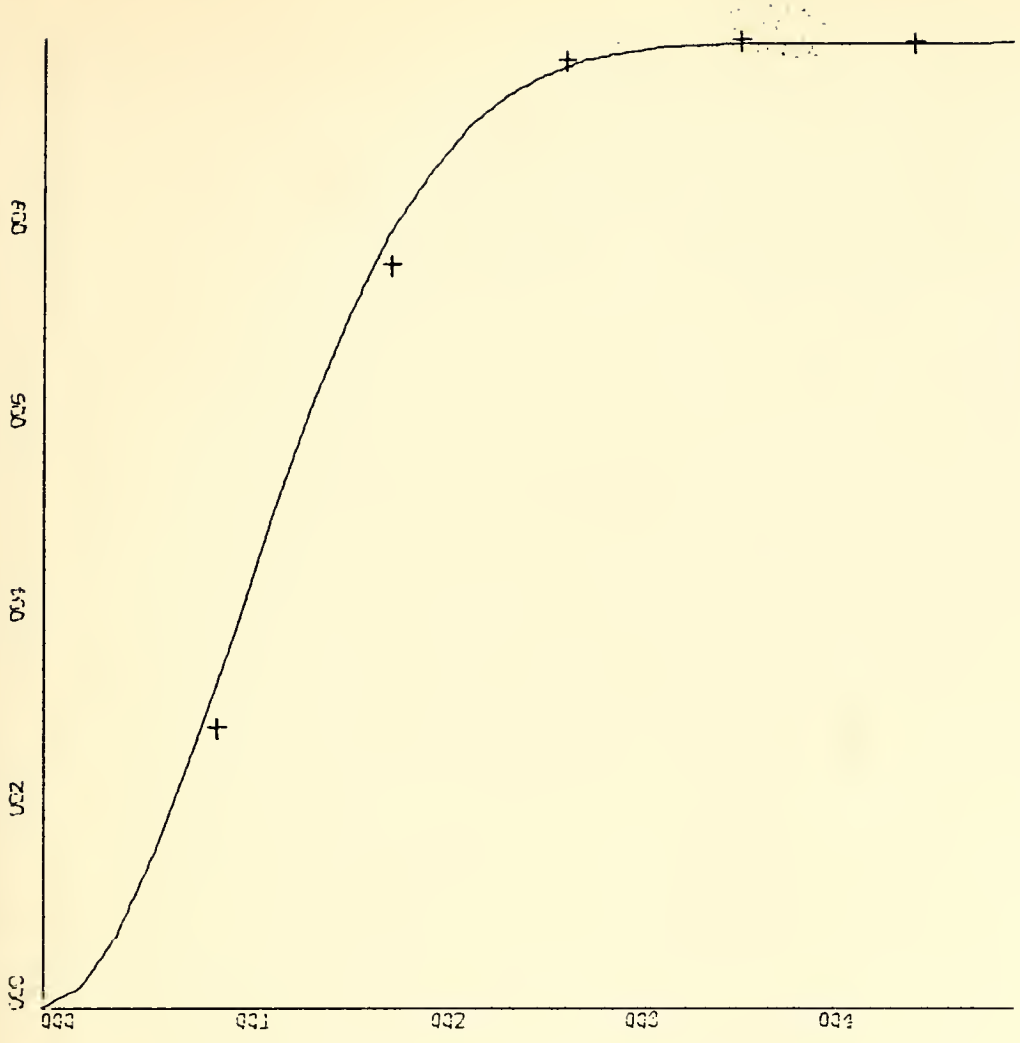
FIGURE 67



FOR DELTA T = 0.8 TFIN = 5.2

T	Q(N)
0.8	0.24242
1.6	0.68588
2.4	0.94231
3.2	0.99899
4.0	1.00011
4.8	0.99998

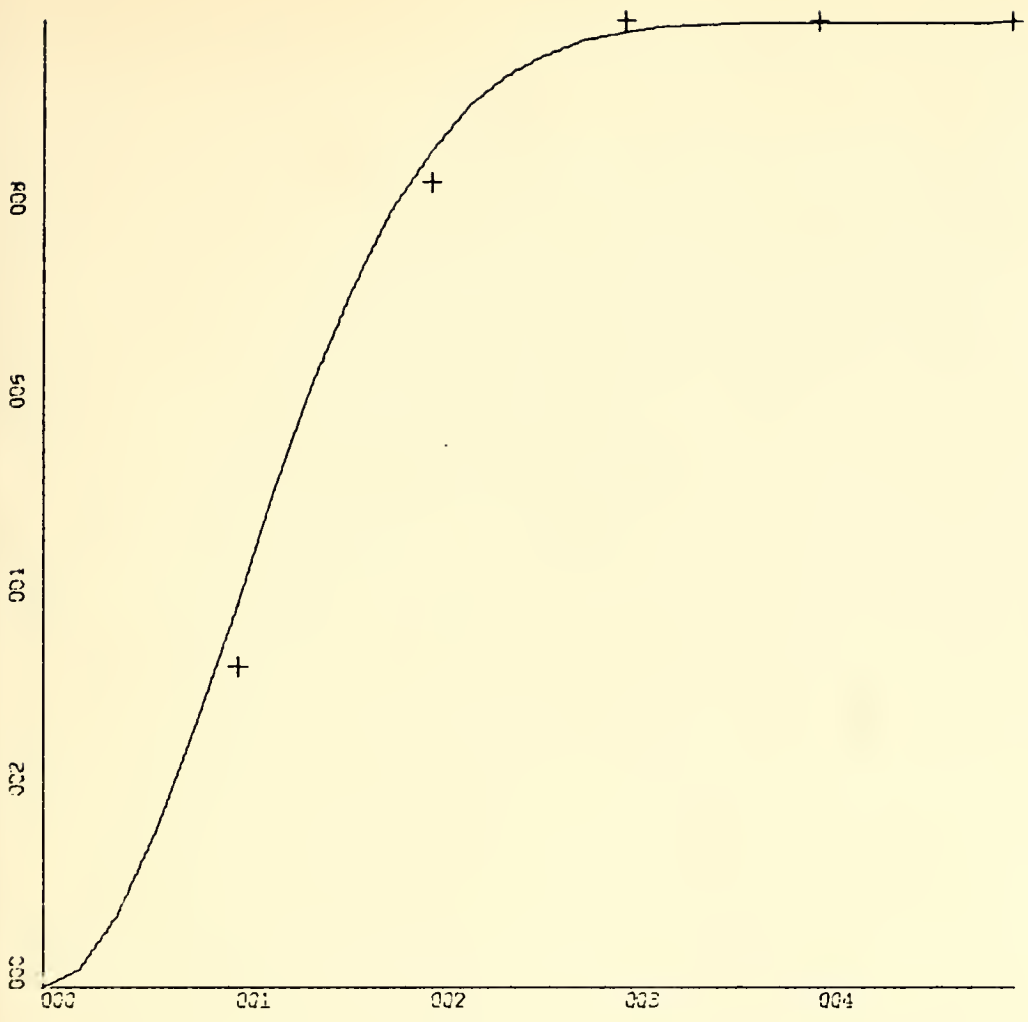
FIGURE 68



FOR DELTA T = 0.9 TFIN = 5.2

T	Q(N)
0.9	0.28826
1.8	0.76603
2.7	0.97993
3.6	1.00165
4.5	0.99966

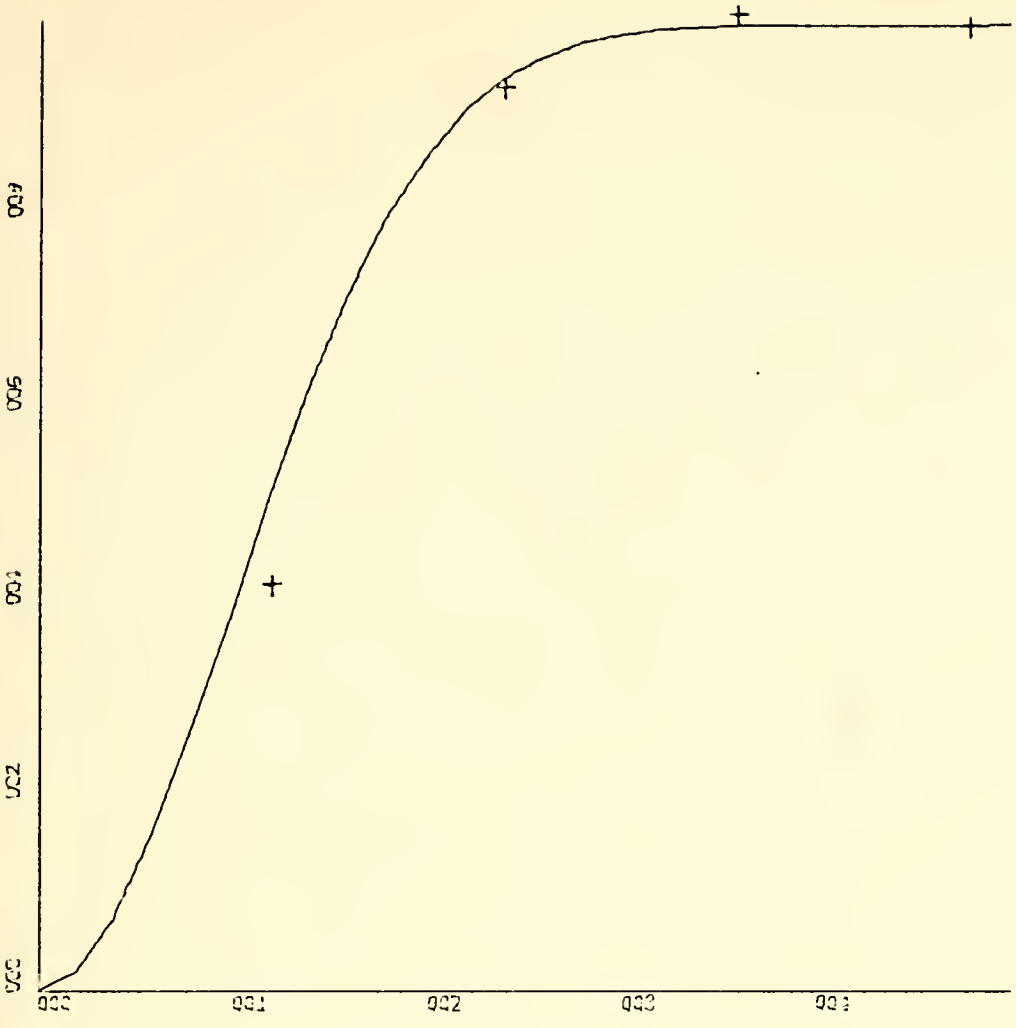
FIGURE 69



FOR DELTA T = 1.0 TFIN = 5.2

T	Q(N)
1.0	0.33333
2.0	0.83333
3.0	1.00000
4.0	1.00000
5.0	1.00000

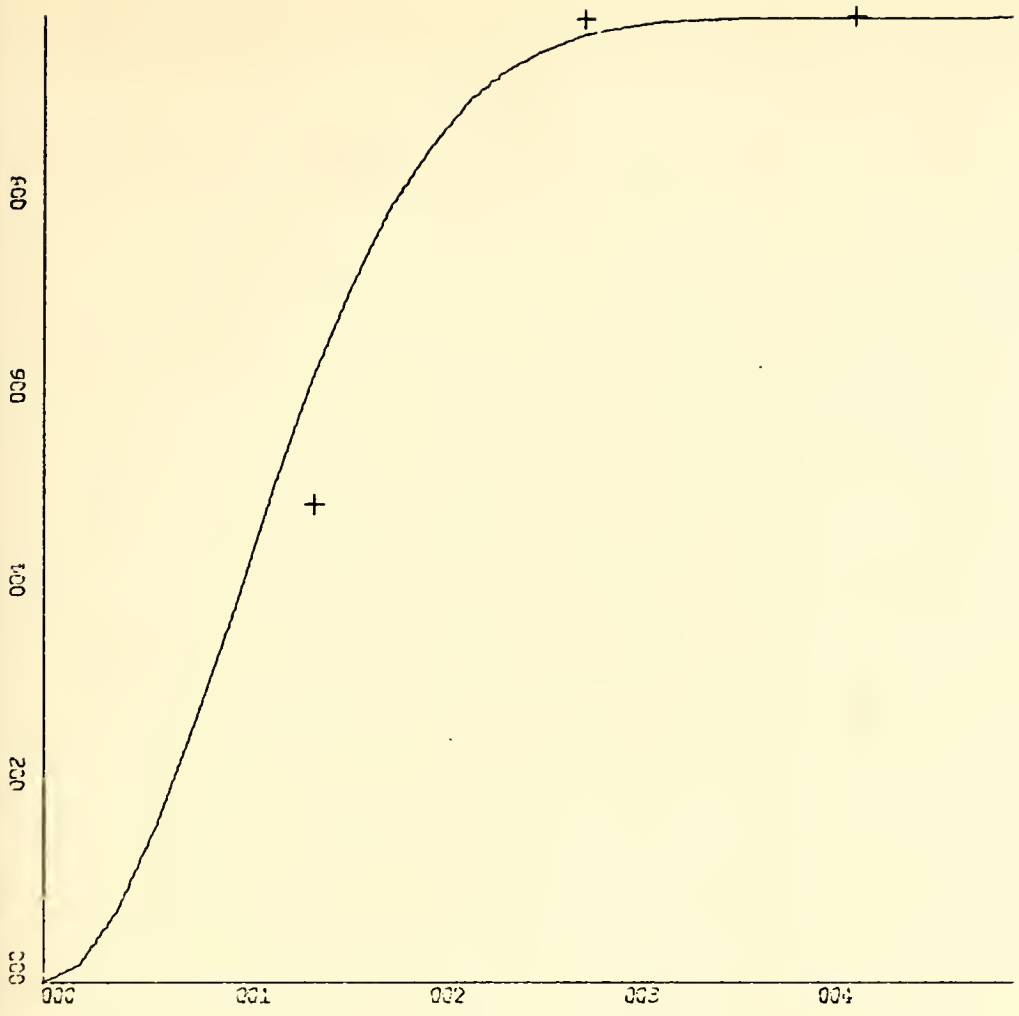
FIGURE 70



FOR DELTA T = 1.2 TFIN = 5.2

T	Q(N)
1.2	0.41860
2.4	0.93328
3.6	1.00929
4.8	0.99722

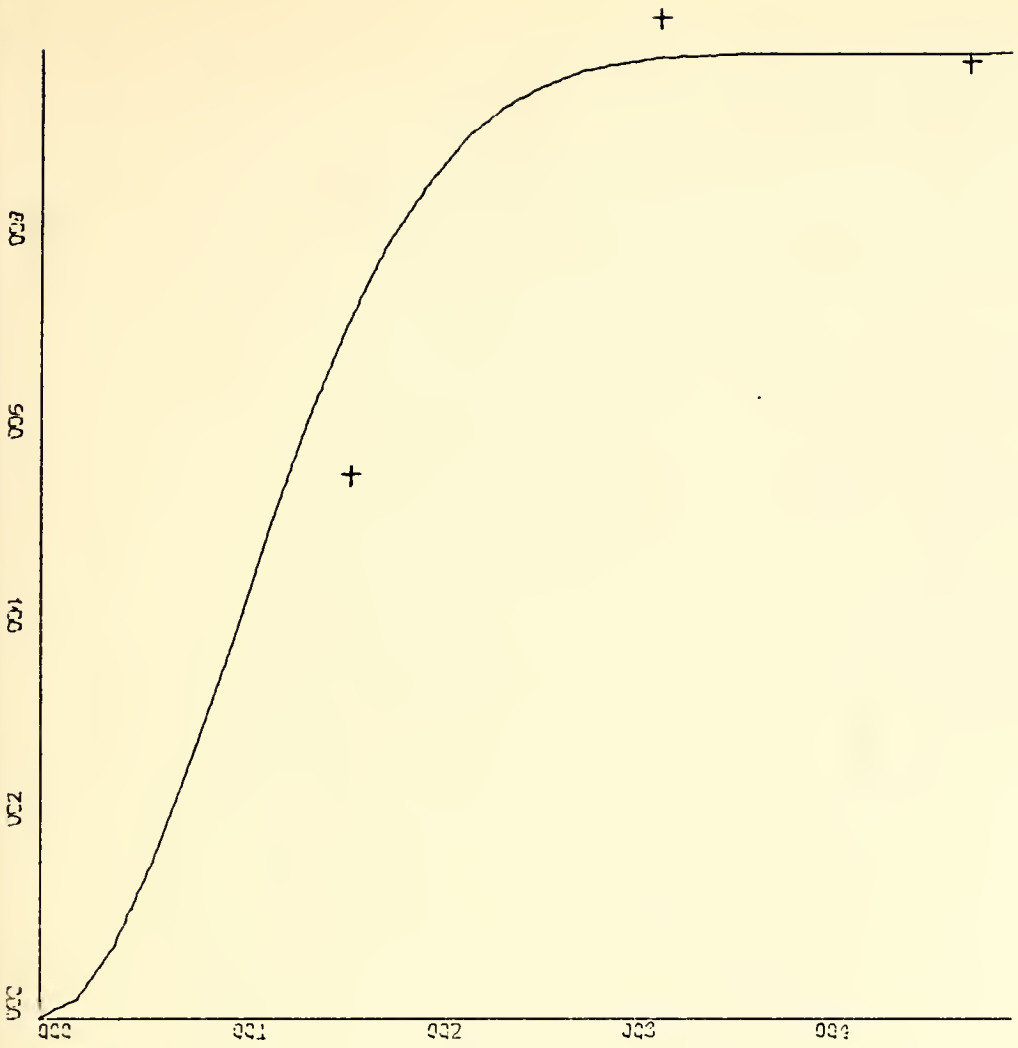
FIGURE 71



FOR DELTA T = 1.4 TFIN = 5.2

T	Q(N)
1.4	0.49495
2.8	0.99659
4.2	1.00083

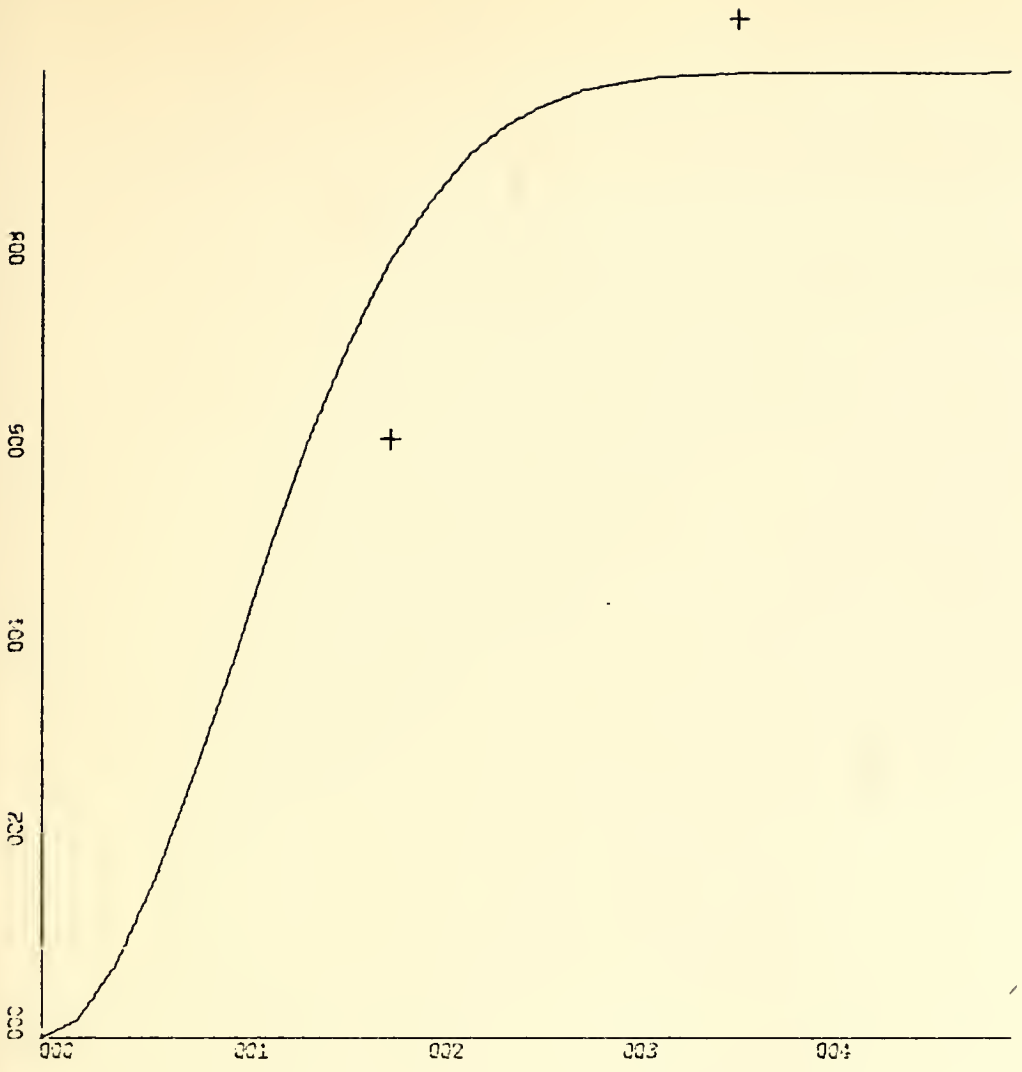
FIGURE 72



FOR DELTA T = 1.6 TFIN = 5.2

T	Q(N)
1.6	0.56140
3.2	1.03450
4.8	0.98888

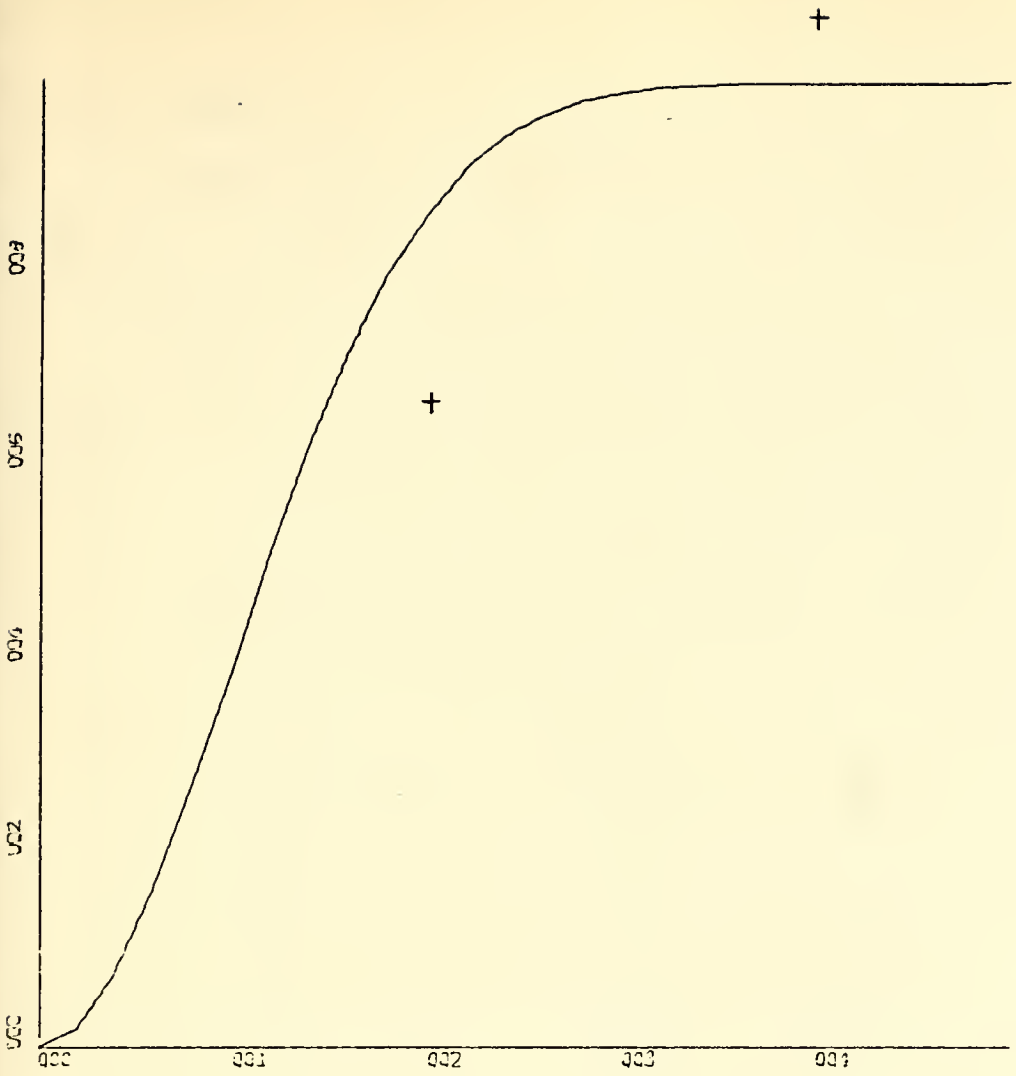
FIGURE 73



FOR DELTA T = 1.8 TFIN = 5.2

T	Q(N)
1.8	0.61832
3.6	1.05581

FIGURE 74



FOR DELTA T = 2.0 TFIN = 5.2

T	Q(N)
2.0	0.66667
4.0	1.06667

FIGURE 75

2. Case 6 — $\dot{Y} + t^2Y = t$

Calculations follow exactly Case 4 except for $P(t) = t^2$. In the division steps, substitution of P for its value t^2 must be done in this case.

Solutions for several time iterations follow (Figs. 76 to 89). Computer solution is in Program 6 (page 111).

FOR DELTA T = 0.2 TFIN - 5.2

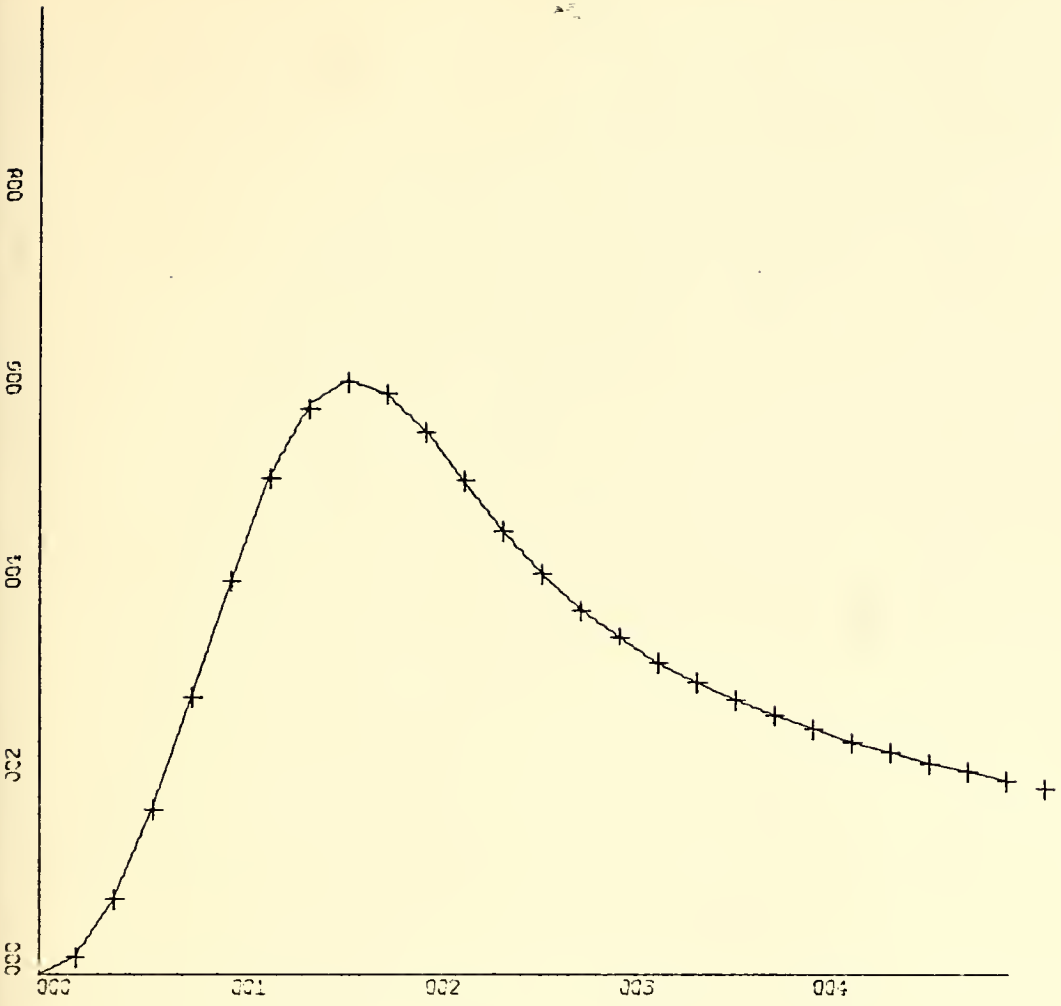
T	Q(N)	PR
0.2	0.01992	0.01997
0.4	0.07858	0.07898
0.6	0.17116	0.17243
0.8	0.28666	0.28923
1.0	0.40755	0.41144
1.2	0.51294	0.51742
1.4	0.58451	0.58837
1.6	0.61301	0.61523
1.8	0.60127	0.60163
2.0	0.56176	0.56095
2.2	0.51015	0.50914
2.4	0.45891	0.45829
2.6	0.41443	0.41425
2.8	0.37796	0.37801
3.0	0.34824	0.34833
3.2	0.32354	0.32360
3.4	0.30253	0.30257
3.6	0.28433	0.28436
3.8	0.26835	0.26837
4.0	0.25418	0.25420
4.2	0.24150	0.24152
4.4	0.23008	0.23009
4.6	0.21973	0.21974
4.8	0.21029	0.21030
5.0	0.20166	0.20167
5.2	0.19372	

Equation (46) is repeated here for convenience:

$$y_A^*(z) = \frac{T^2z^{-1} + T^2z^{-2}}{(2+PT) - (6+PT)z^{-1} + (6-PT)z^{-2} + (PT-2)z^{-3}}$$

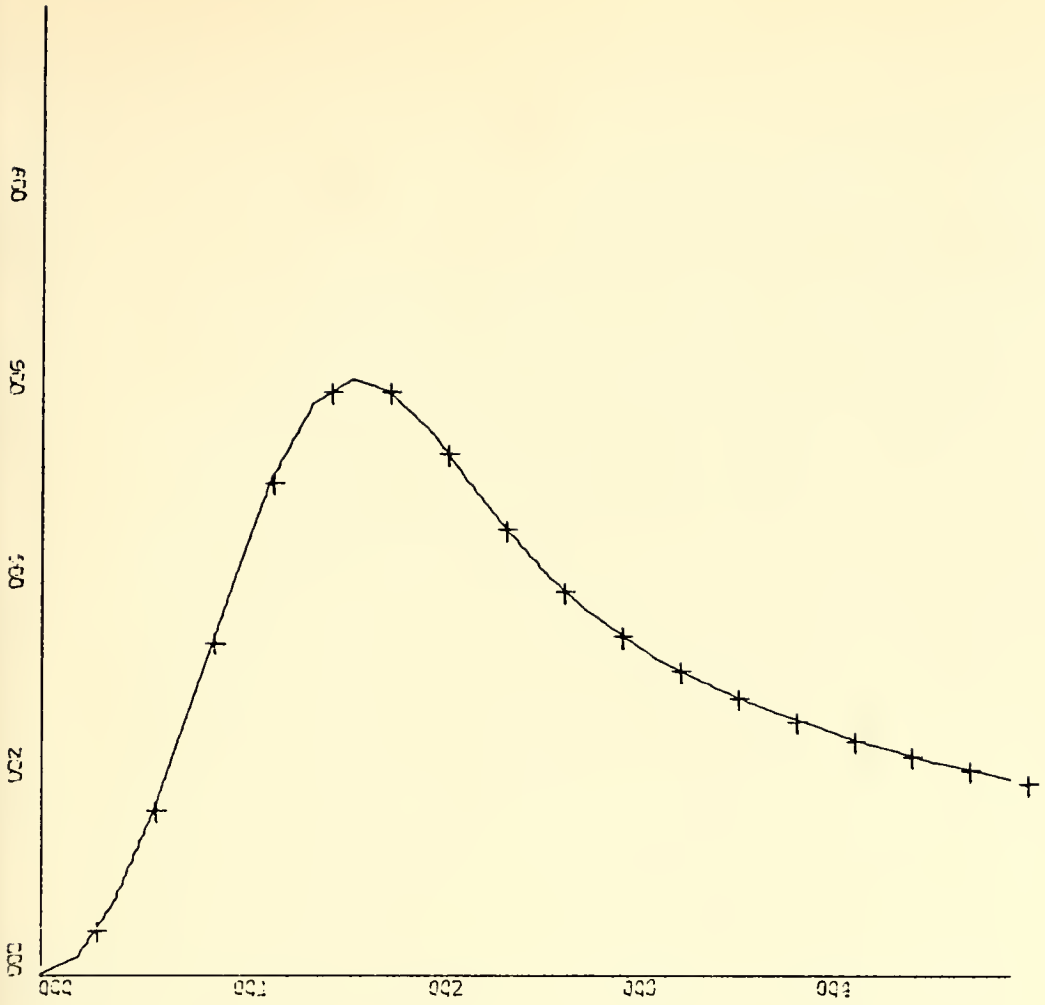
```
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
C   Q(T)=RAMP  &  P(T)=T**2
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
      INTEGER *4ITB(12)/12*0/
      REAL *4RTB(28)/28*0.0/
      DIMENSION A(100), Q(100), X(100), XX(100), PR(100)
      ITB(3) = 5
      ITB(4) = 5
      WRITE (6,8)
C
      DO 1 I=1,26
      READ (5,7) XX(I),PR(I)
      WRITE (6,7) XX(I),PR(I)
    1 CONTINUE
C
    2 READ (5,7,END=4) TD,TFIN
      WRITE (6,5) TD,TFIN
      WRITE (6,6)
      X(1) = 0.0
      M = TFIN/TD
      A(1) = TD**2
      A(2) = TD**2
      A(3) = 0.0
      A(4) = 0.0
C
      DO 3 N=1,M
      T = N*TD
      P = T**2
      F1 = 2.0+P*TD
      F2 = -(6.0+P*TD)
      F3 = 6.0-P*TD
      F4 = (P*TD)-2.0
C
      Q(N) = A(N)/F1
C
      A(N+1) = A(N+1)-(Q(N)*F2)
      A(N+2) = A(N+2)-(Q(N)*F3)
      A(N+3) = A(N+3)-(Q(N)*F4)
      A(N+4) = 0.0
      WRITE (6,7) T,Q(N)
      X(N) = T
    3 CONTINUE
C
      ITB(1) = 1
      ITB(2) = 0
      RTB(6) = TD
      ITB(12) = 1
      CALL DRAWP (26,XX,PR,ITB,RTB)
      ITB(1) = 3
      ITB(2) = 2
      CALL DRAWP (M,X,Q,ITB,RTB)
      GO TO 2
    4 STOP
C
    5 FORMAT ('1',' FOR DELTA T=',F6.2,4X,'TFIN=',F4.1)
    6 FORMAT ('  T=',7X,' Q(N)=')
    7 FORMAT (2F10.5)
    8 FORMAT ('  XX=',7X,'    PR=')
      END
```

PROGRAM 6



x-scale = 1.0 units/inch
y-scale = 0.2 units/inch
FOR DELTA T = 0.2

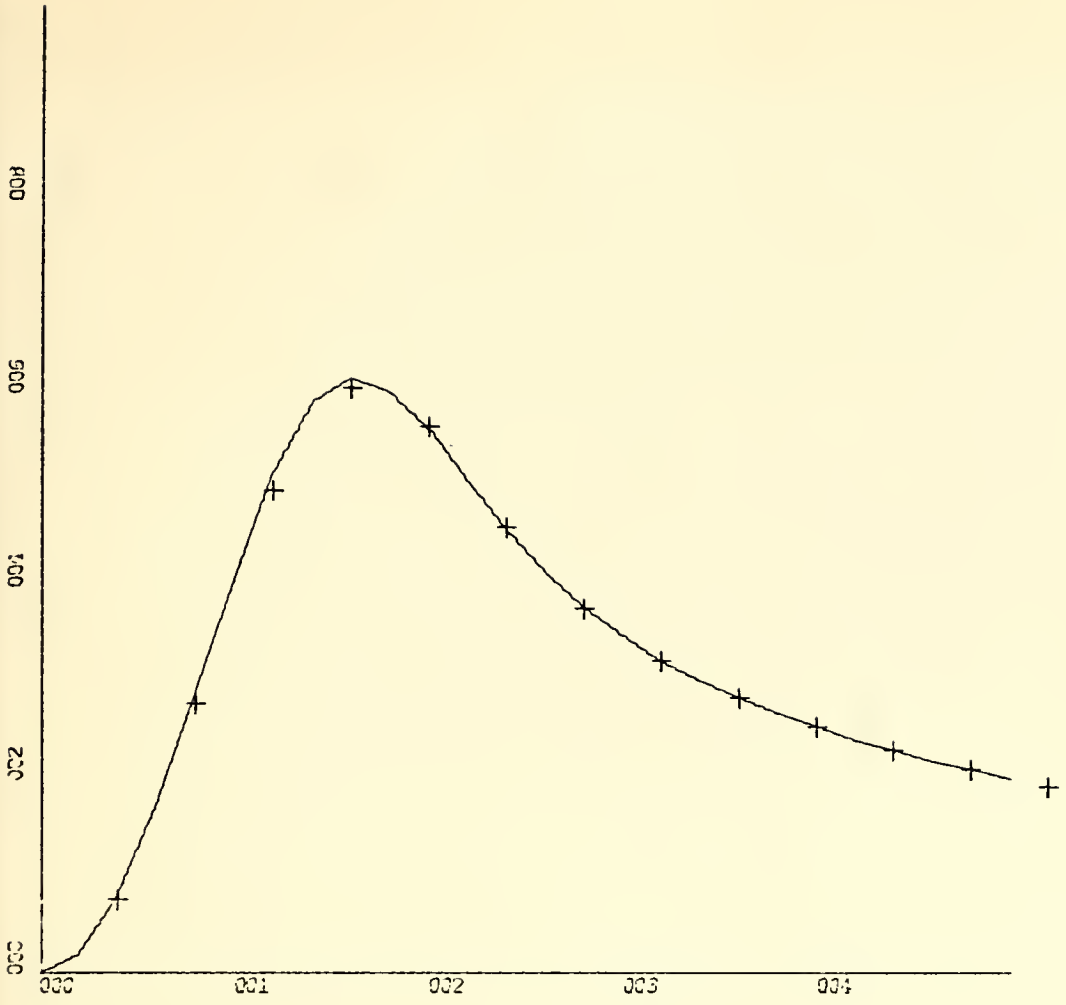
FIGURE 76



FOR DELTA T = 0.3 TFIN = 5.2

T	Q(N)
0.3	0.04440
0.6	0.16964
0.9	0.34372
1.2	0.50736
1.5	0.60020
1.8	0.60070
2.1	0.53792
2.4	0.45981
2.7	0.39529
3.0	0.34810
3.3	0.31257
3.6	0.28430
3.9	0.26104
4.2	0.24149
4.5	0.22477
4.8	0.21028
5.1	0.19760

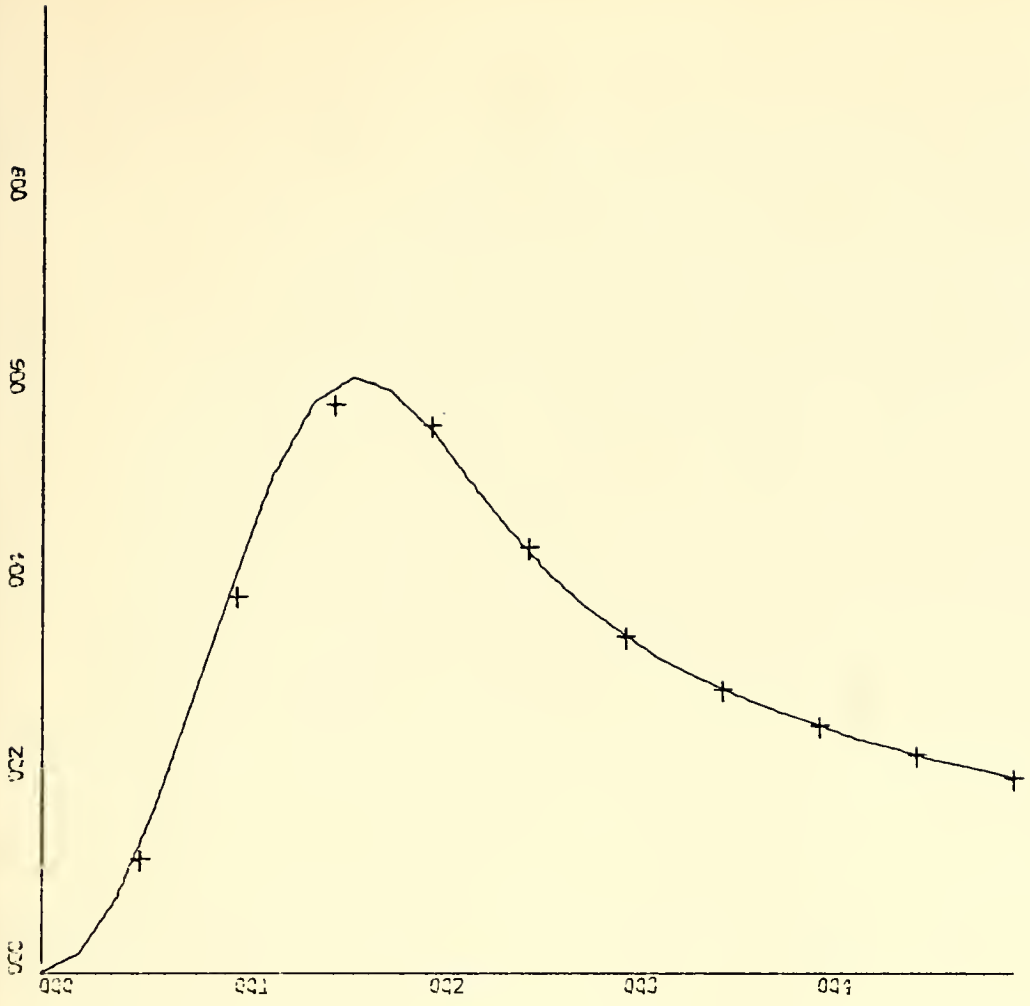
FIGURE 77



FOR DELTA T = 0.4 TFIN = 5.2

T	Q(N)
0.4	0.07752
0.8	0.27929
1.2	0.49964
1.6	0.60565
2.0	0.56420
2.4	0.46136
2.8	0.37768
3.2	0.32332
3.6	0.28429
4.0	0.25415
4.4	0.23007
4.8	0.21028
5.2	0.19371

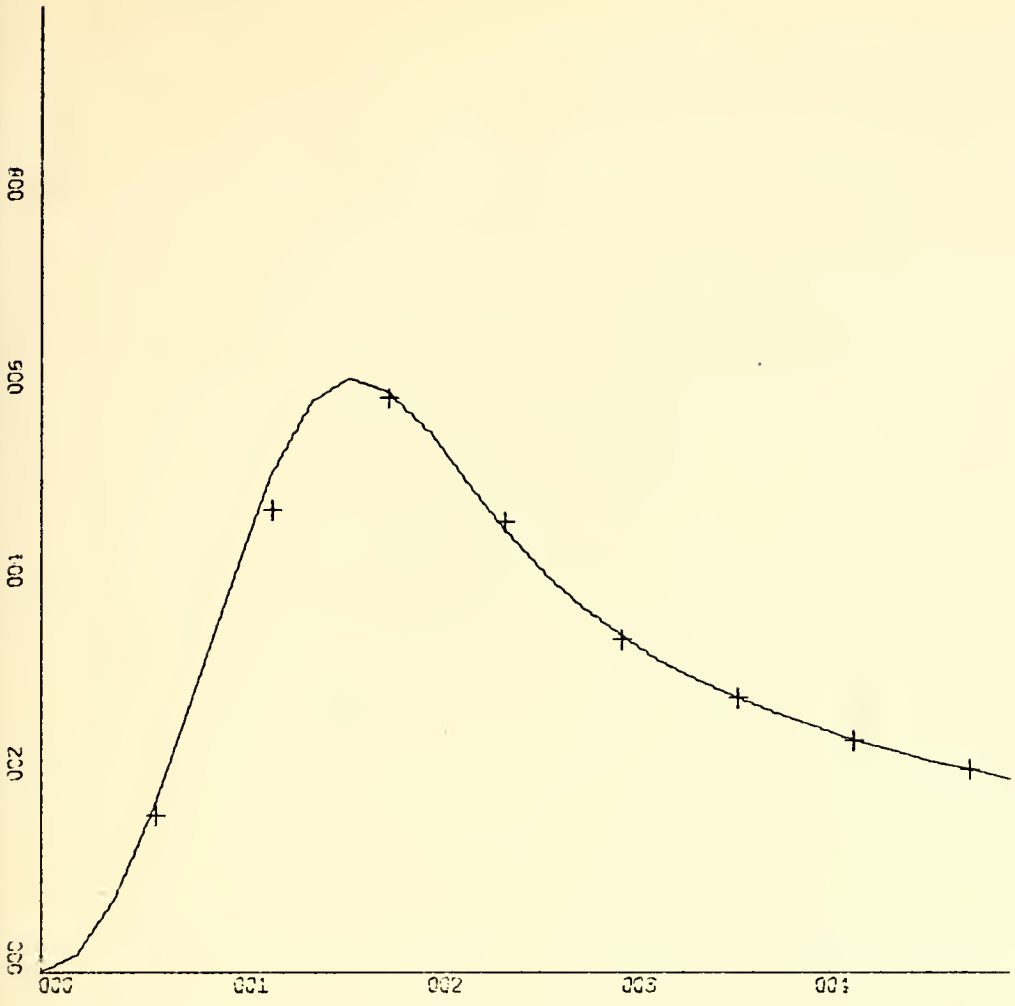
FIGURE 78



FOR DELTA T = 0.5 TFIN = 5.2

T	Q(N)
0.5	0.11765
1.0	0.38824
1.5	0.58635
2.0	0.56576
3.0	0.34709
3.5	0.29320
4.0	0.25405
4.5	0.22480
5.0	0.20162

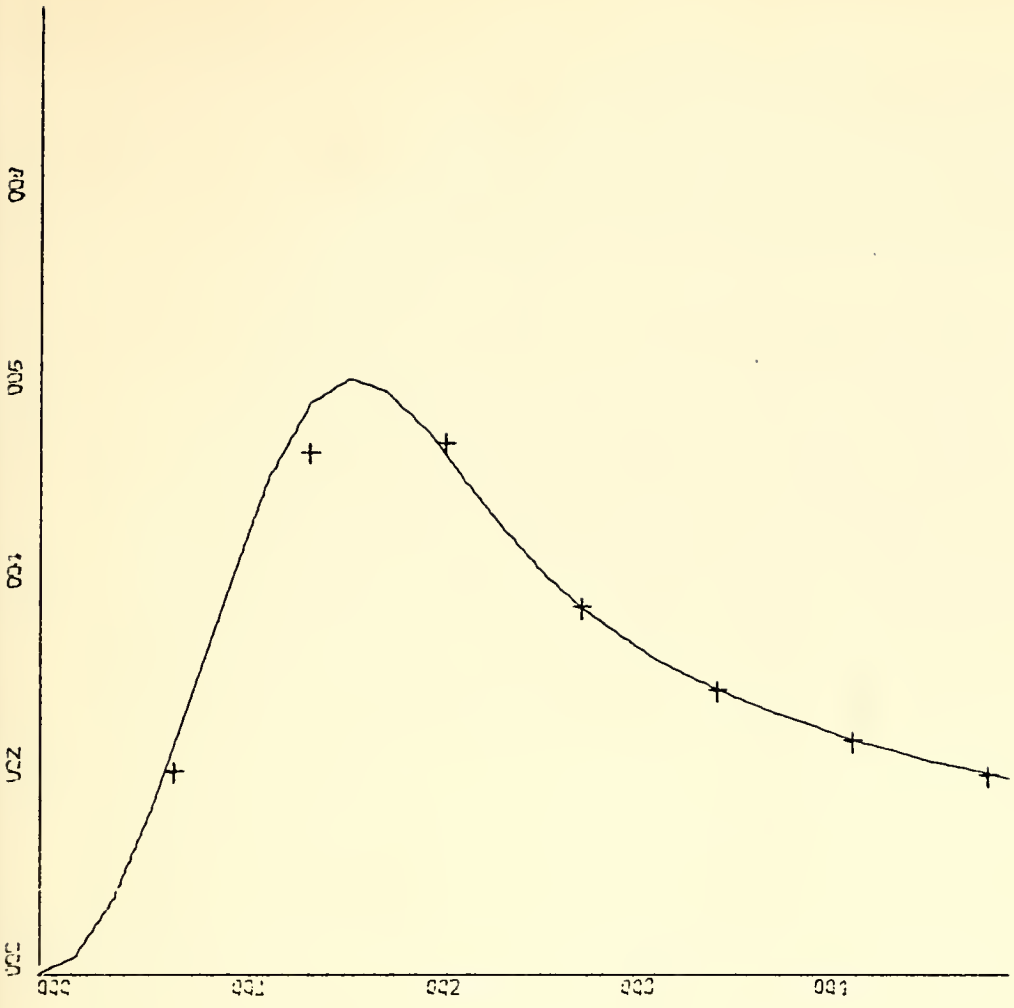
FIGURE 79



FOR DELTA T = 0.6 TFIN = 5.2

T	Q(N)
0.6	0.16245
1.2	0.47829
1.8	0.59415
2.4	0.46797
3.0	0.34576
3.6	0.28482
4.2	0.24117
4.8	0.21043

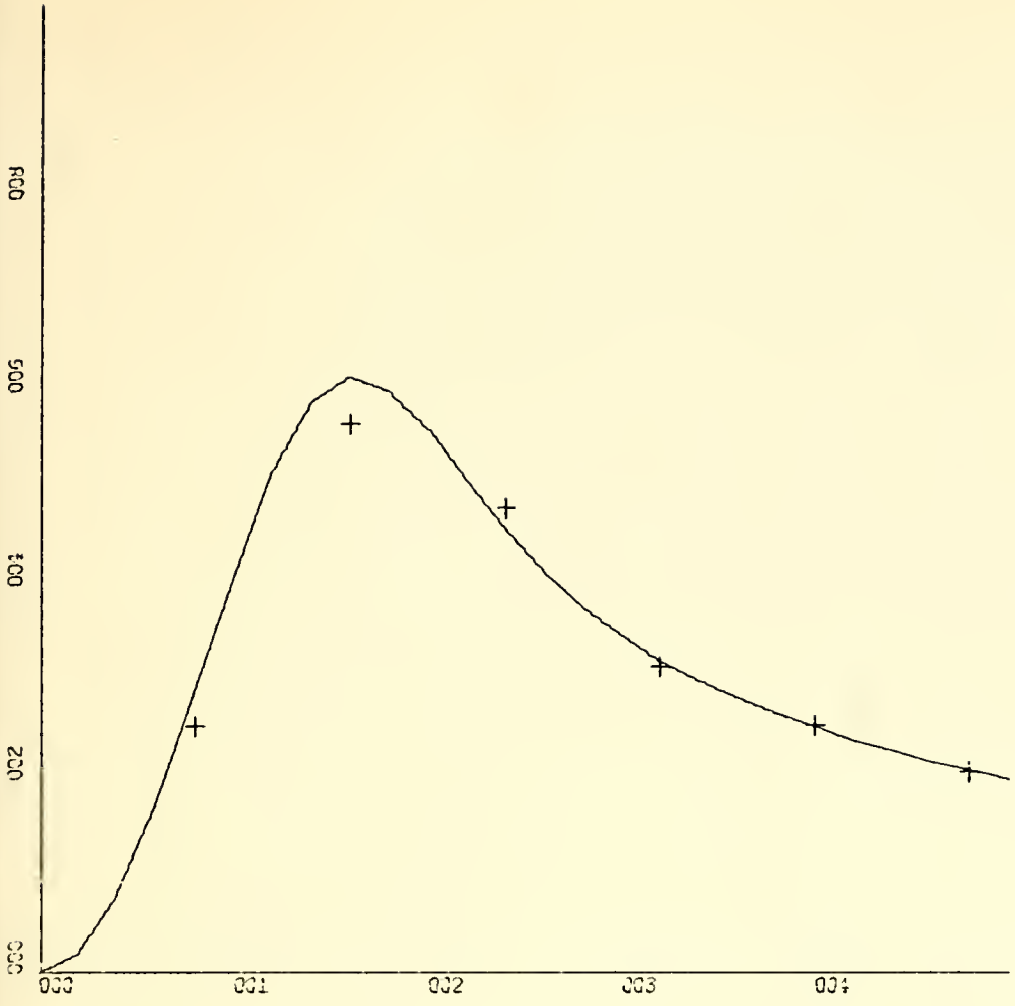
FIGURE 80



FOR DELTA T = 0.7 TFIN = 5.2

T	Q(N)
0.7	0.20913
1.4	0.53871
2.1	0.54812
2.8	0.37850
3.5	0.29218
4.2	0.24177
4.9	0.20568

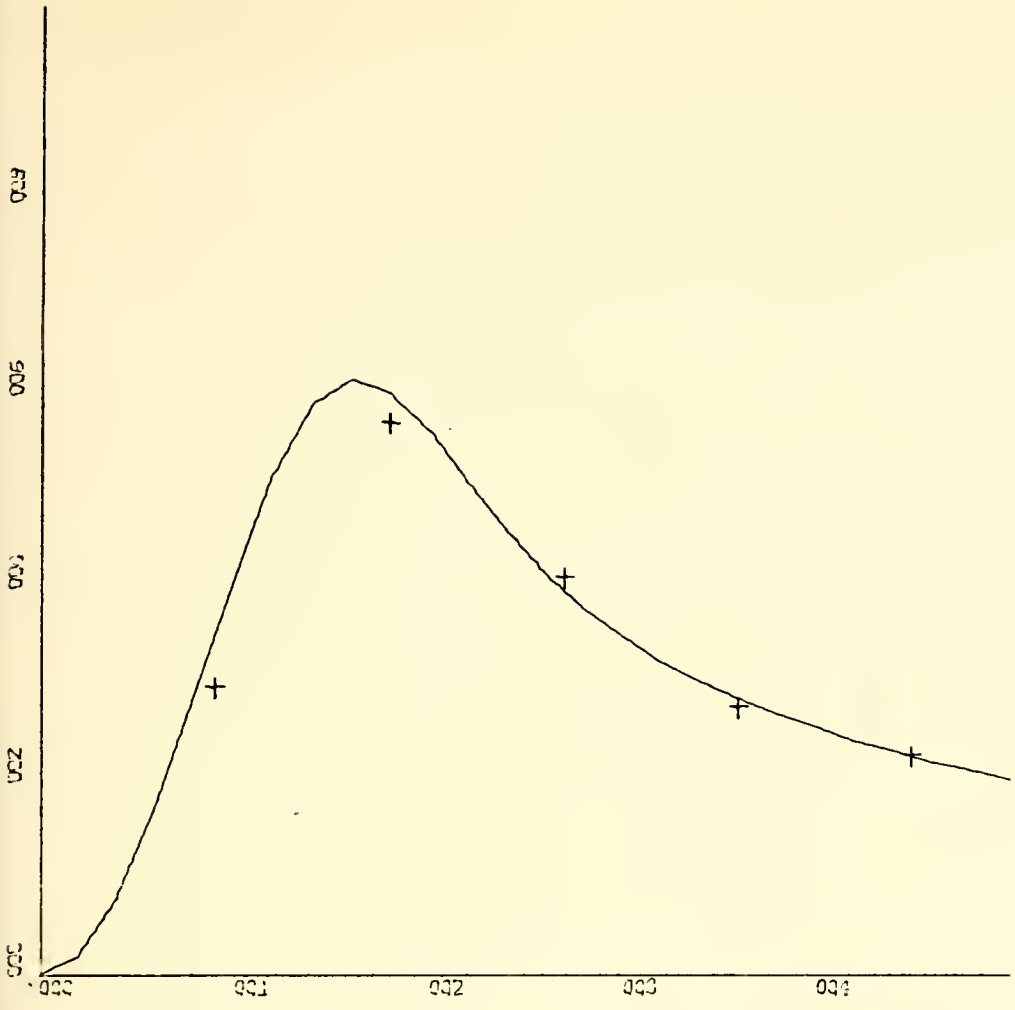
FIGURE 81



FOR DELTA T = 0.8 TFIN = 5.2

T	Q(N)
0.8	0.25478
1.6	0.56796
2.4	0.48014
3.2	0.31670
4.0	0.25669
4.8	0.20888

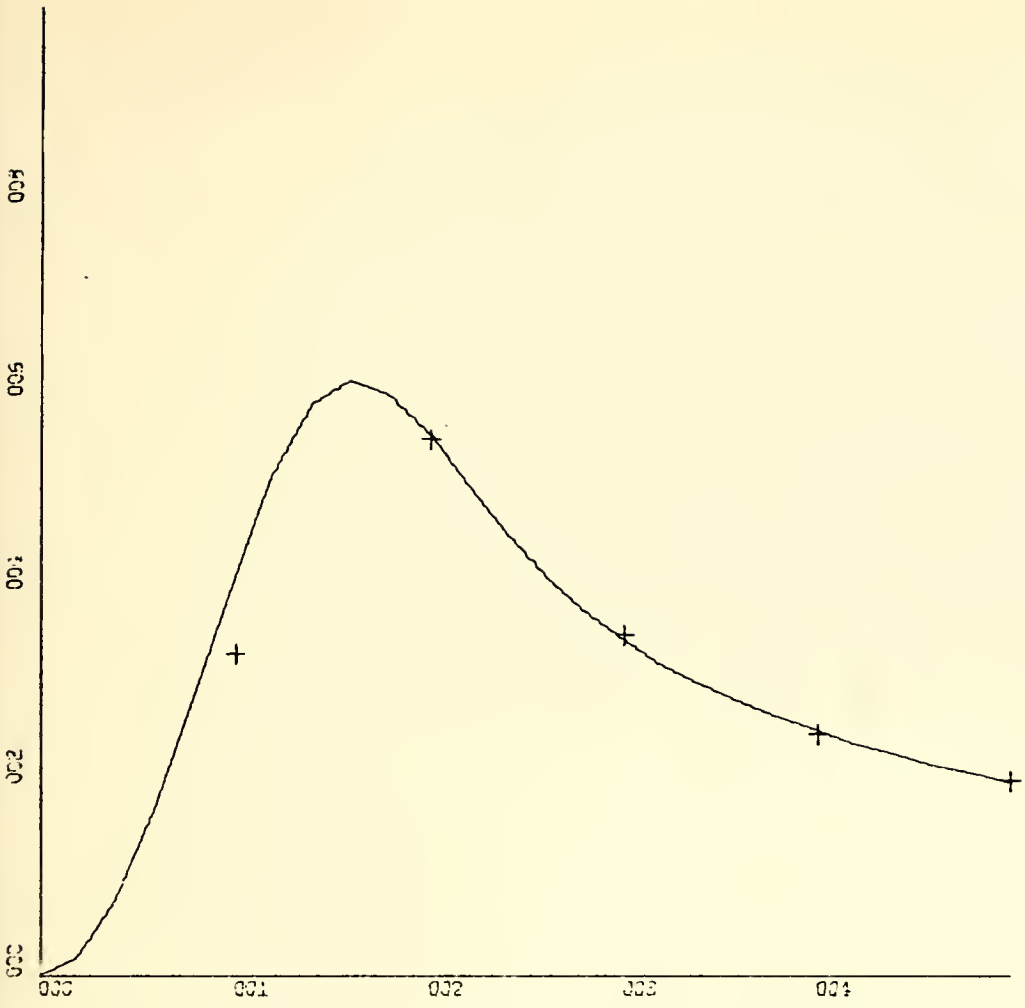
FIGURE 82



FOR DELTA T = 0.9 TFIN = 5.2

T	Q(N)
0.9	0.29681
1.8	0.57104
2.7	0.41198
3.6	0.27744
4.5	0.22788

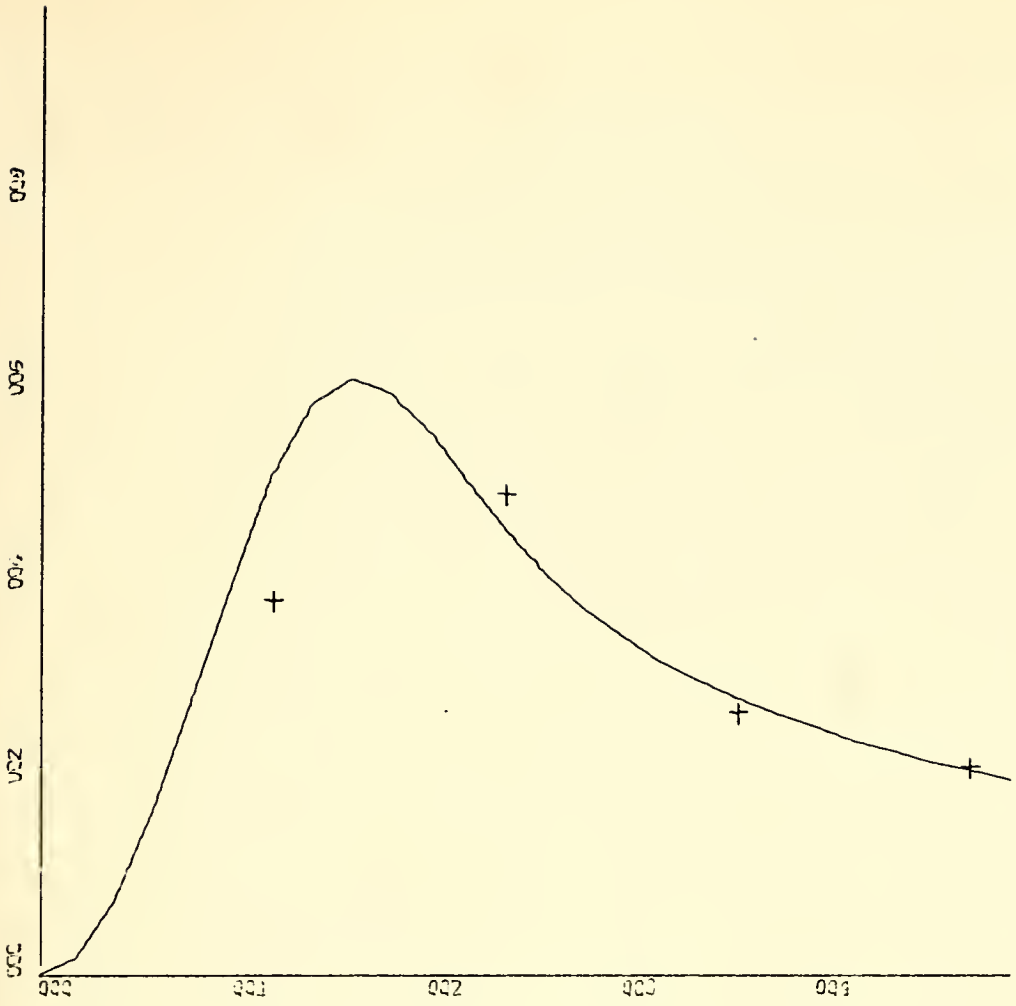
FIGURE 83



FOR DELTA T = 1.0 TFIN = 5.2

T	Q(N)
1.0	0.33333
2.0	0.55556
3.0	0.35354
4.0	0.25140
5.0	0.20298

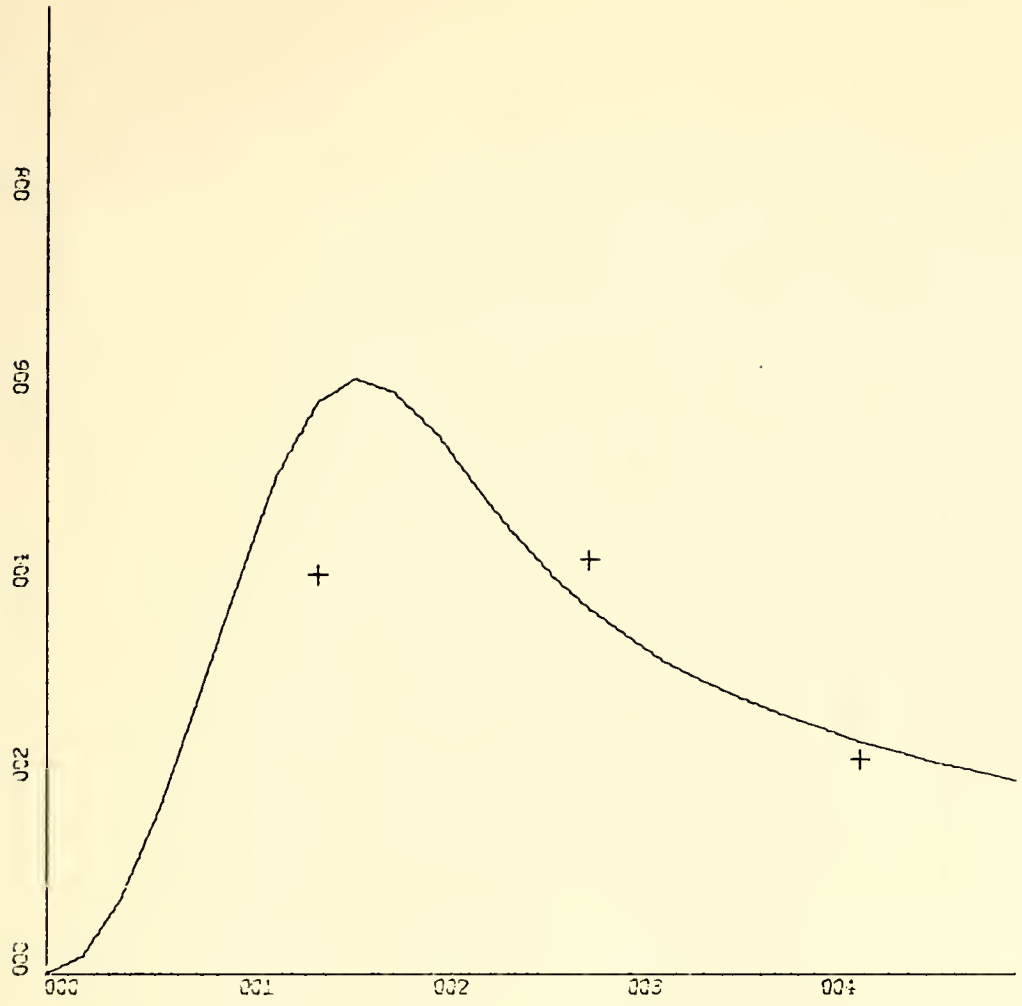
FIGURE 84



FOR DELTA T = 1.2 TFIN = 5.2

T	Q(N)
1.2	0.38627
2.4	0.49653
3.6	0.27125
4.8	0.21600

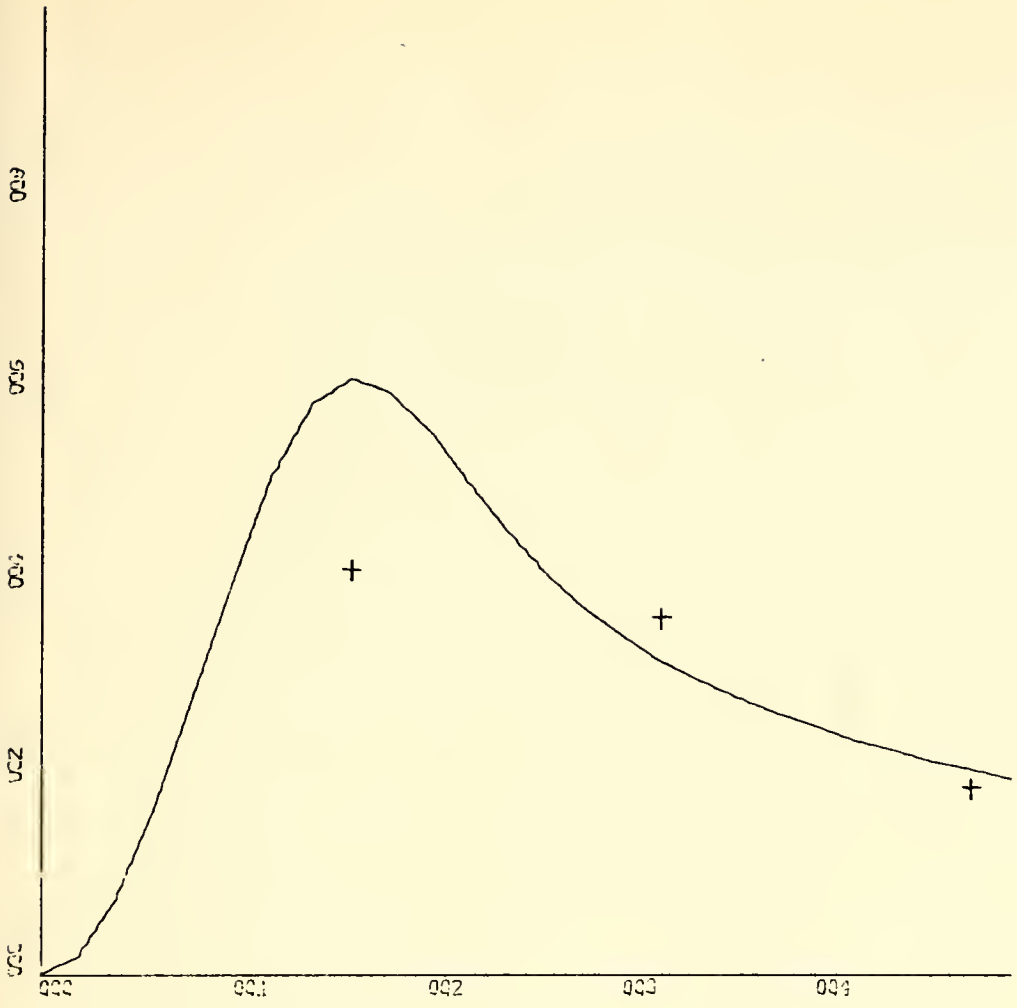
FIGURE 85



FOR DELTA T = 1.4 TFIN = 5.2

T	Q(N)
1.4	0.41315
2.8	0.42946
4.2	0.22270

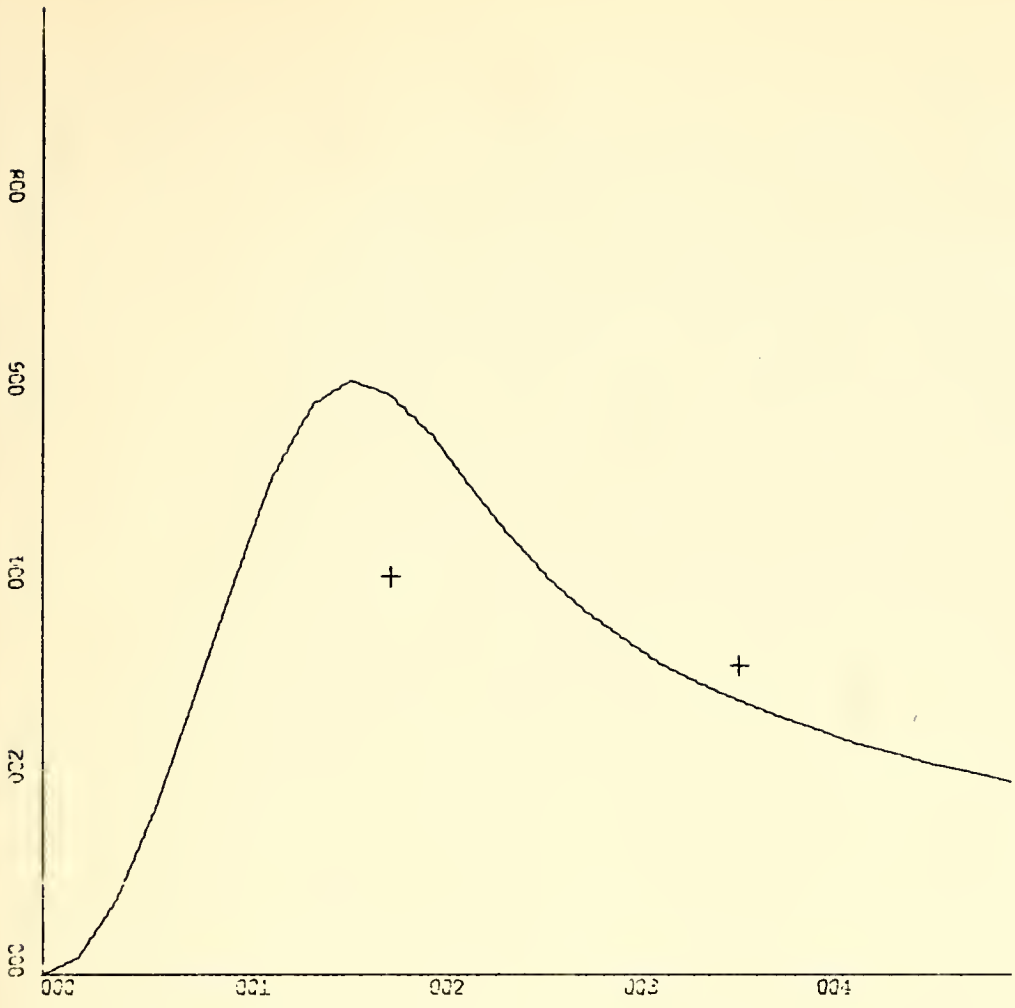
FIGURE 86



FOR DELTA T = 1.6 TFIN = 5.2

T	Q(N)
1.6	0.41995
3.2	0.36988
4.8	0.19246

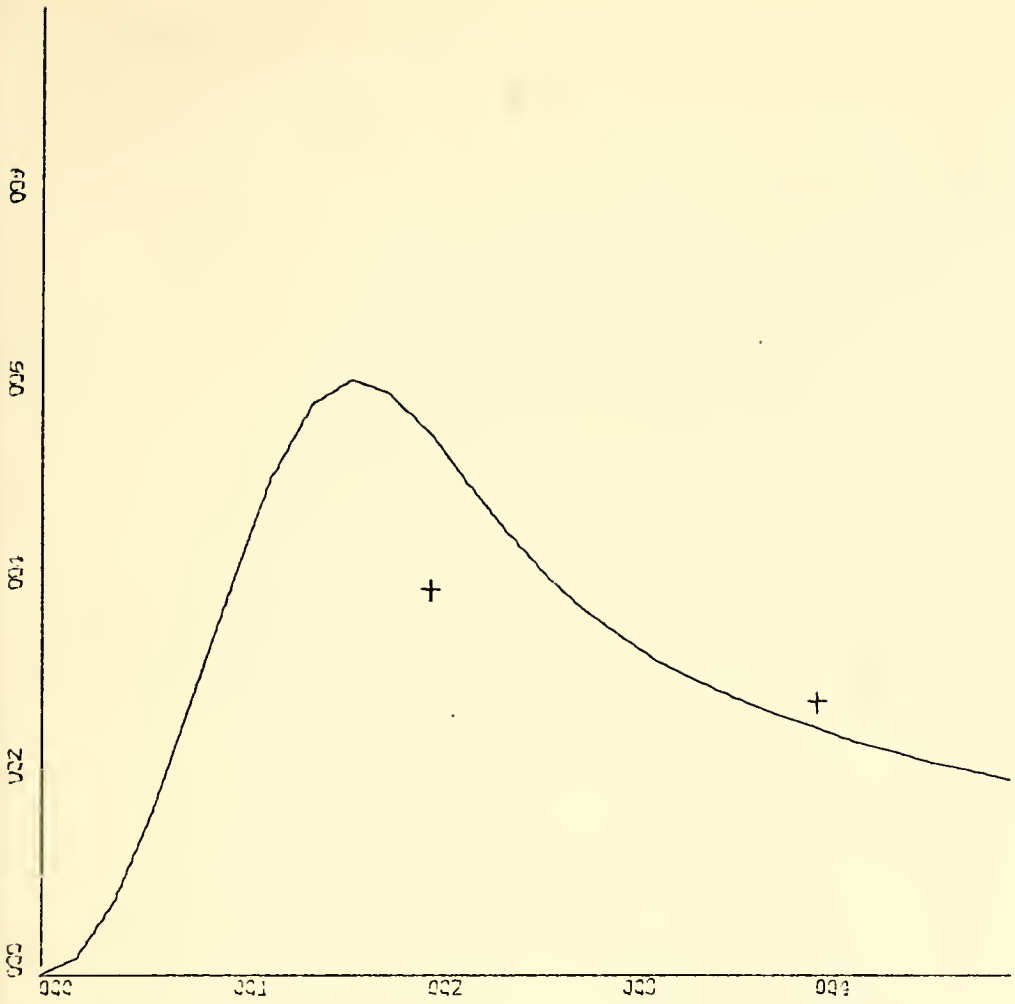
FIGURE 87



FOR DELTA T = 1.8 TFIN = 5.2

T	Q(N)
1.8	0.41369
3.6	0.32118

FIGURE 88



FOR DELTA T = 2.0 TFIN = 5.2

T	Q(N)
2.0	0.40000
4.0	0.28235

FIGURE 89

D. CASE 7 – $\dot{Y} + \sqrt{t}\,Y = t^2$

Following the procedure as in II C:

$$\frac{dY}{dt} + PY = t^2 \qquad P = \sqrt{t}$$

The Laplace transform in powers of s^{-k}:

$$sy + Py = 2/s^3 \qquad y = \frac{2s^{-4}}{1 + Ps^{-1}}$$

Substituting the z-forms of Table I into the expression above yields:

$$y_A^*(z) = \frac{\frac{T^4}{6} \cdot \frac{z^{-1} + 4z^{-2} + z^{-3}}{(1 - z^{-1})^4} - \frac{T^4}{720}}{1 + P\frac{T}{2} \cdot \frac{(1 + z)^{-1}}{(1 - z)^{-1}}}$$

Dividing by T and rearranging in crescent powers of z^{-1}:

$$y_A^*(z) = \frac{-T^3 + (124T^3)z^{-1} + (474T^3)z^{-2} + 124T^3z^{-3} = T^3z^{-4}}{360+180PT-(1440+360PT)z^{-1}+2160z^{-2}-(1440-360PT)z^{-3}+ \ldots \ldots + (360-180PT)z^{-4}} \qquad (47)$$

Choosing for first value for T = 0.2, and substituting in the previous equation, the denominator is divided into the numerator, after the value of P is replaced in the expression above for the corresponding time.

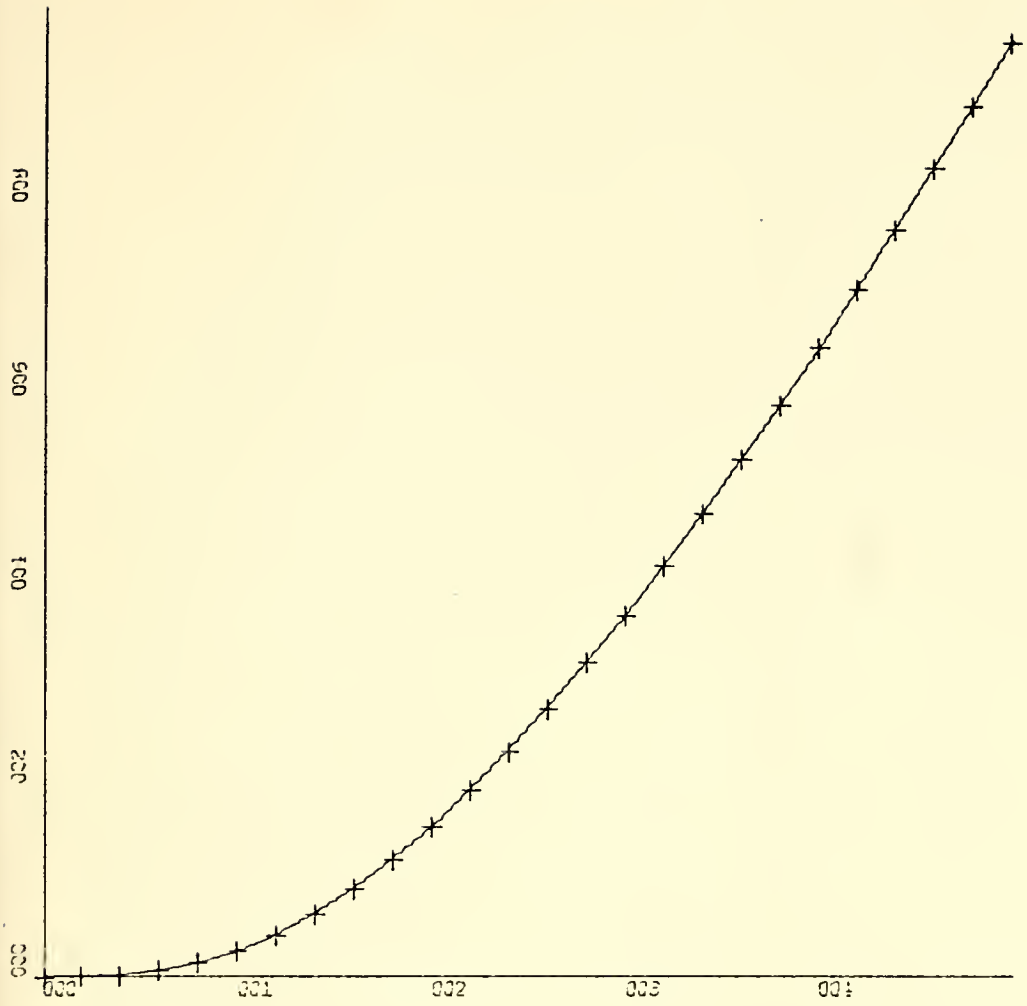
The results and the plots for the precise and approximated solution follow, with iteration times from 0.2 sec. to 2.0 sec. (Figs. 90 to 103). Computer algorithm for solution of this problem is in Program 7 (page 128).

DELTA T = 0.2 TFIN = 5.2

T	Q(N)	PR
0.0	-0.00002	0.0
0.2	0.00255	0.00261
0.4	0.01985	0.02018
0.6	0.06428	0.06511
0.8	0.14500	0.14648
1.0	0.26791	0.27012
1.2	0.43602	0.43897
1.4	0.65001	0.65363
1.6	0.90881	0.91298
1.8	1.21016	1.21470
2.0	1.55110	1.55590
2.2	1.92838	1.93330
2.4	2.33869	2.34370
2.6	2.77893	2.78380
2.8	3.24627	3.25100
3.0	3.73821	3.74280
3.2	4.25262	4.25700
3.4	4.78767	4.79180
3.6	5.34186	5.34590
3.8	5.91392	5.91780
4.0	6.50283	6.50670
4.2	7.10775	7.11160
4.4	7.72795	7.73200
4.6	8.36285	8.36720
4.8	9.01195	9.01680
5.0	9.67481	9.68030

```
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
C  Q(T) = T**2  &  P(T) =SQRT(T)
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
      INTEGER *4ITB(12)/12*0/
      REAL *4RTB(28)/28*0.0/
      DIMENSION A(100), Q(100), X(100), XX(100), PR(100)
      ITB(3) = 5
      ITB(4) = 5
      WRITE (6,8)
C
      DO 1 I=1,26
      READ (5,5) XX(I),PR(I)
      WRITE (6,5) XX(I),PR(I)
    1 CONTINUE
C
    2 READ (5,5,END=4) TD,TFIN
      WRITE (6,6) TD,TFIN
      WRITE (6,7)
      M = TFIN/TD
      T3 = TD**3
      A(1) = -T3
      A(2) = 124.0*T3
      A(3) = 474.0*T3
      A(4) = 124.0*T3
      A(5) = -T3
C
      DO 3 N=1,M
      T = (N-1)*TD
      P = SQRT(T)
      PT = P*TD
      F1 = 360.0+180.0*PT
      F2 = -(1440.0+360.0*PT)
      F3 = 2160.0
      F4 = -(1440.0-360.0*PT)
      F5 = 360.0-180.0*PT
C
      Q(N) = A(N)/F1
C
      A(N+1) = A(N+1)-(Q(N)*F2)
      A(N+2) = A(N+2)-(Q(N)*F3)
      A(N+3) = A(N+3)-(Q(N)*F4)
      A(N+4) = A(N+4)-(Q(N)*F5)
      A(N+5) = 0.0
      WRITE (6,5) T,Q(N)
      X(N) = T
    3 CONTINUE
C
      ITB(1) = 1
      ITB(2) = 0
      ITB(12) = 1
      CALL DRAWP (26,XX,PR,ITB,RTB)
      ITB(1) = 3
      ITB(2) = 2
      CALL DRAWP (M,X,Q,ITB,RTB)
      GO TO 2
    4 STOP
C
    5 FORMAT (2F10.5)
    6 FORMAT ('1','FOR DELTA T=',F6.2,4X,'TFIN=',F4.1)
    7 FORMAT ('  T=',8X,'  Q(N)=')
    8 FORMAT ('   XX=',7X,'   PR=')
      END
```

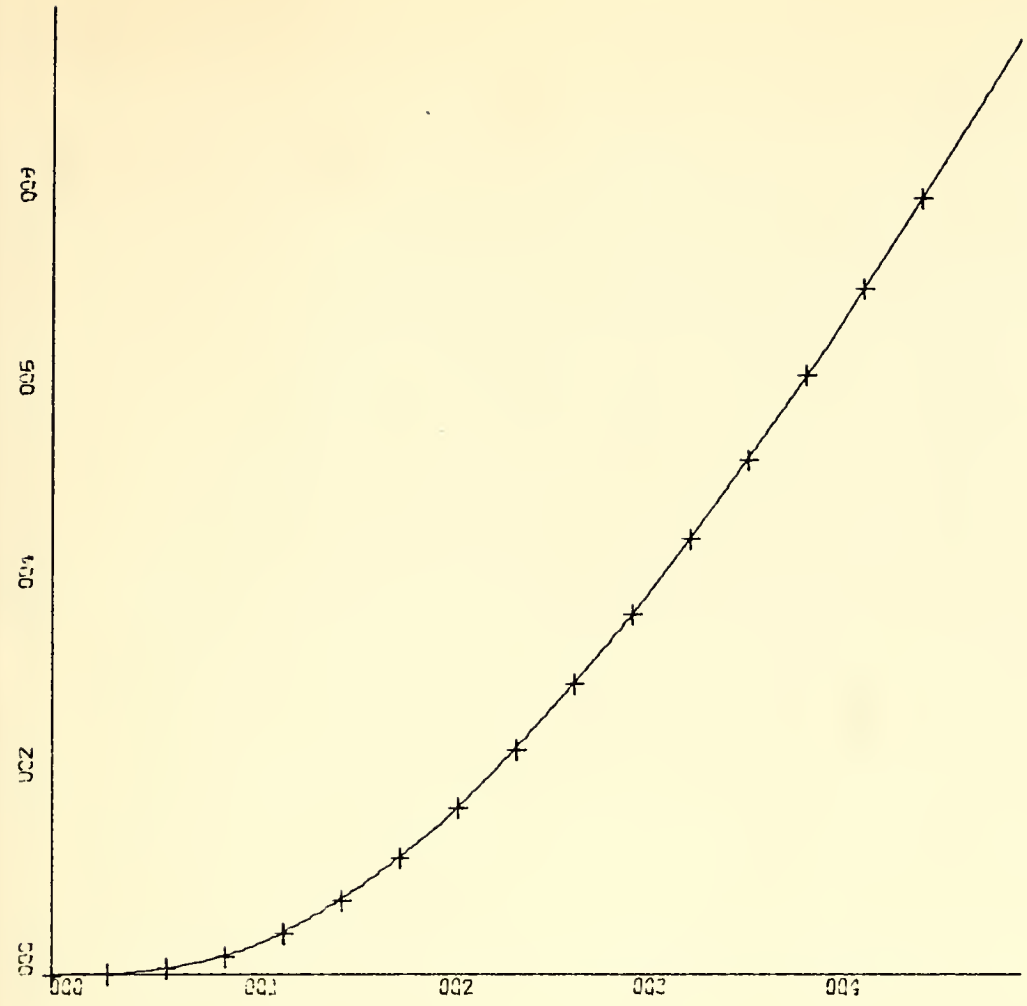
PROGRAM 7



x-scale = 1.0 units/inch
y-scale = 2.0 units/inch

FOR DELTA T = 0.2

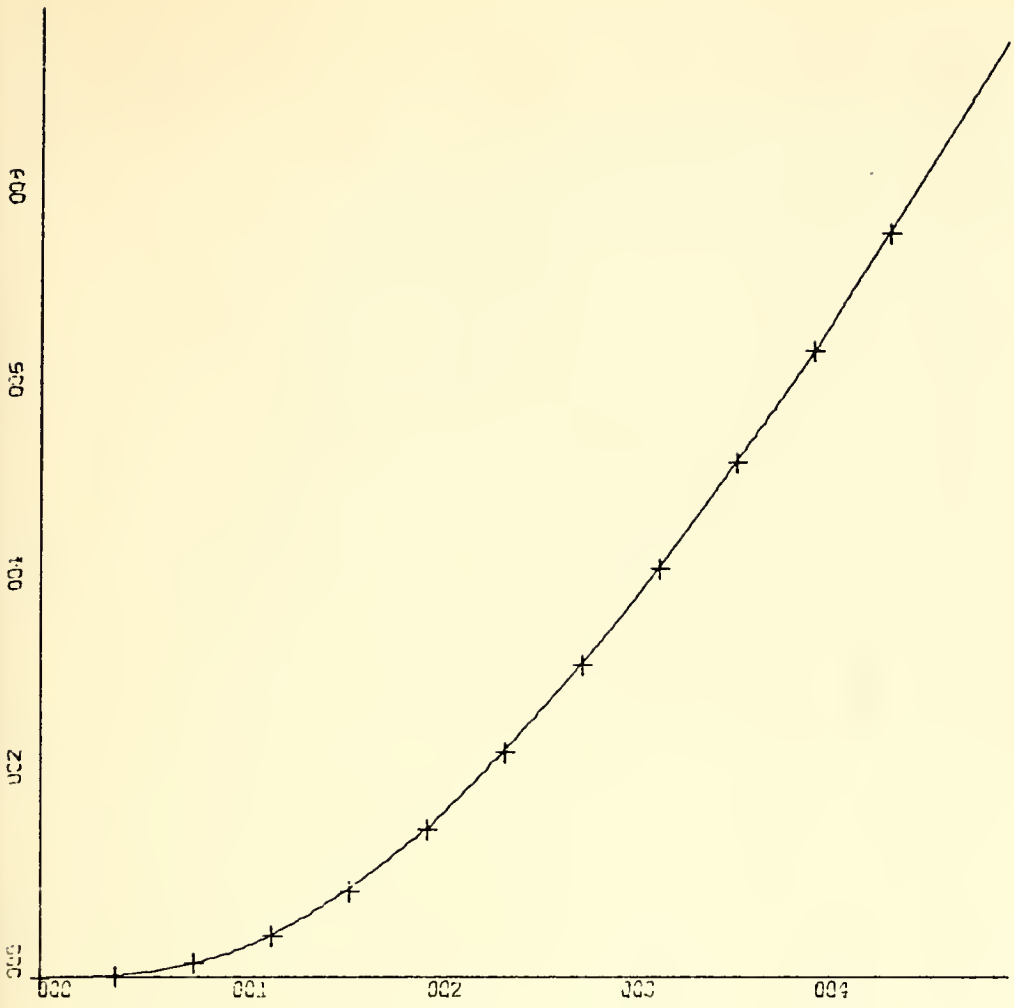
FIGURE 90



FOR DELTA T = 0.3 TFIN = 5.0

T	Q(N)
0.0	-0.00007
0.3	0.00832
0.6	0.06328
0.9	0.19866
1.2	0.43235
1.5	0.76902
1.8	1.20437
2.1	1.72913
2.4	2.33229
2.7	3.00312
3.0	3.73225
3.3	4.51208
3.6	5.33673
3.9	6.20176
4.2	7.10386
4.5	8.04059

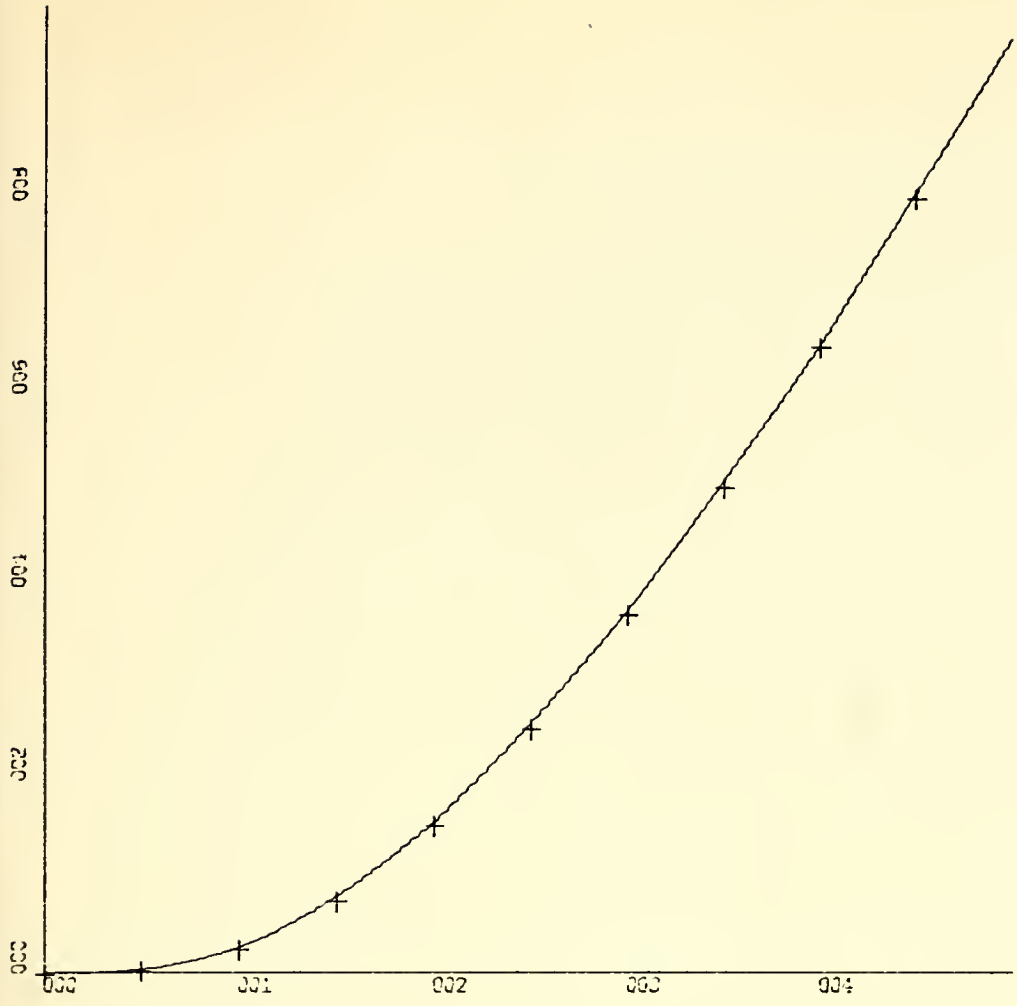
FIGURE 91



FOR DELTA T = 0.4 TFIN = 5.0

T	Q(N)
0.0	-0.00018
0.4	0.01894
0.8	0.14071
1.2	0.42726
1.6	0.89625
2.0	1.53631
2.4	2.32327
2.8	3.23134
3.2	4.23875
3.6	5.32927
4.0	6.49164
4.4	7.71831

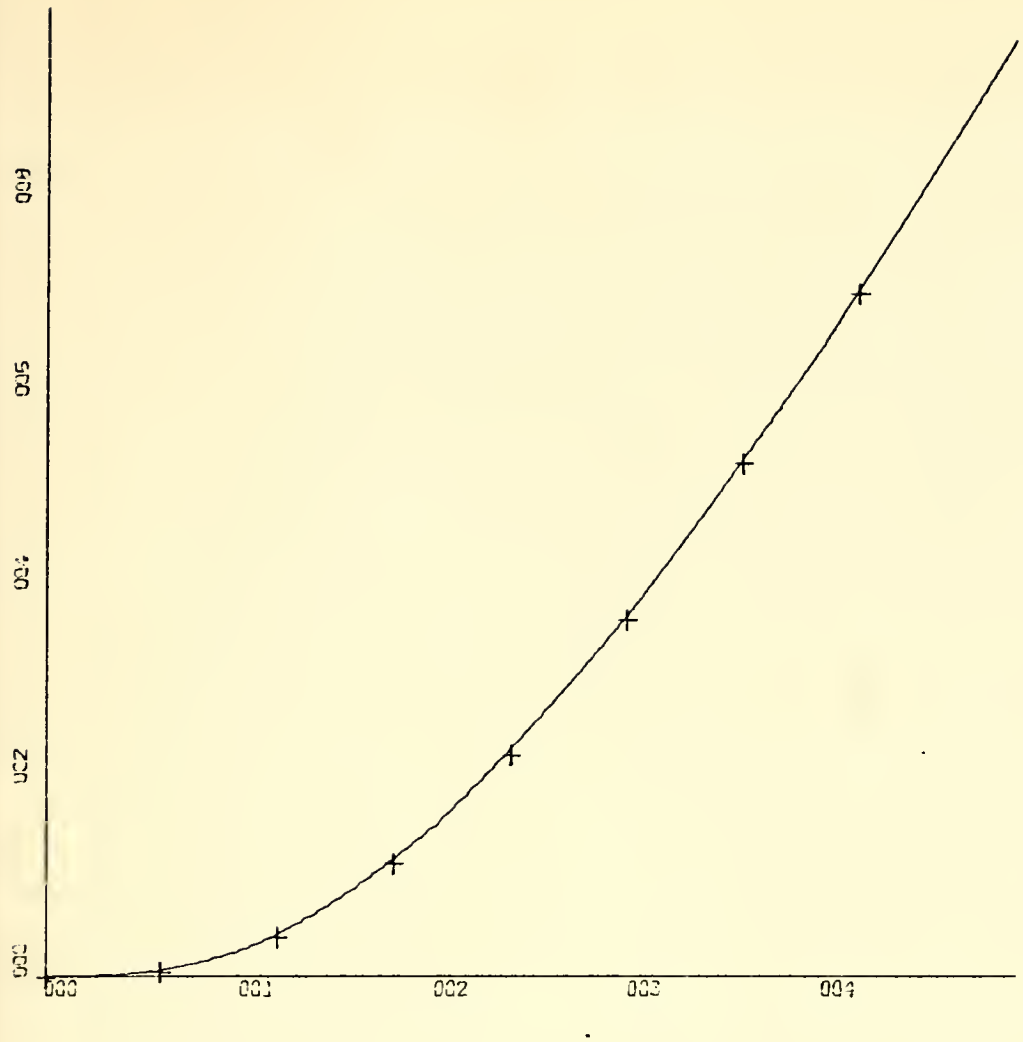
FIGURE 92



FOR DELTA T = 0.5 TFIN = 5.0

T	Q(N)
0.0	-0.00035
0.5	0.03541
1.0	0.25665
1.5	0.75346
2.0	1.52519
2.5	2.52824
3.0	3.71282
3.5	5.03967
4.0	6.48280
4.5	8.02638

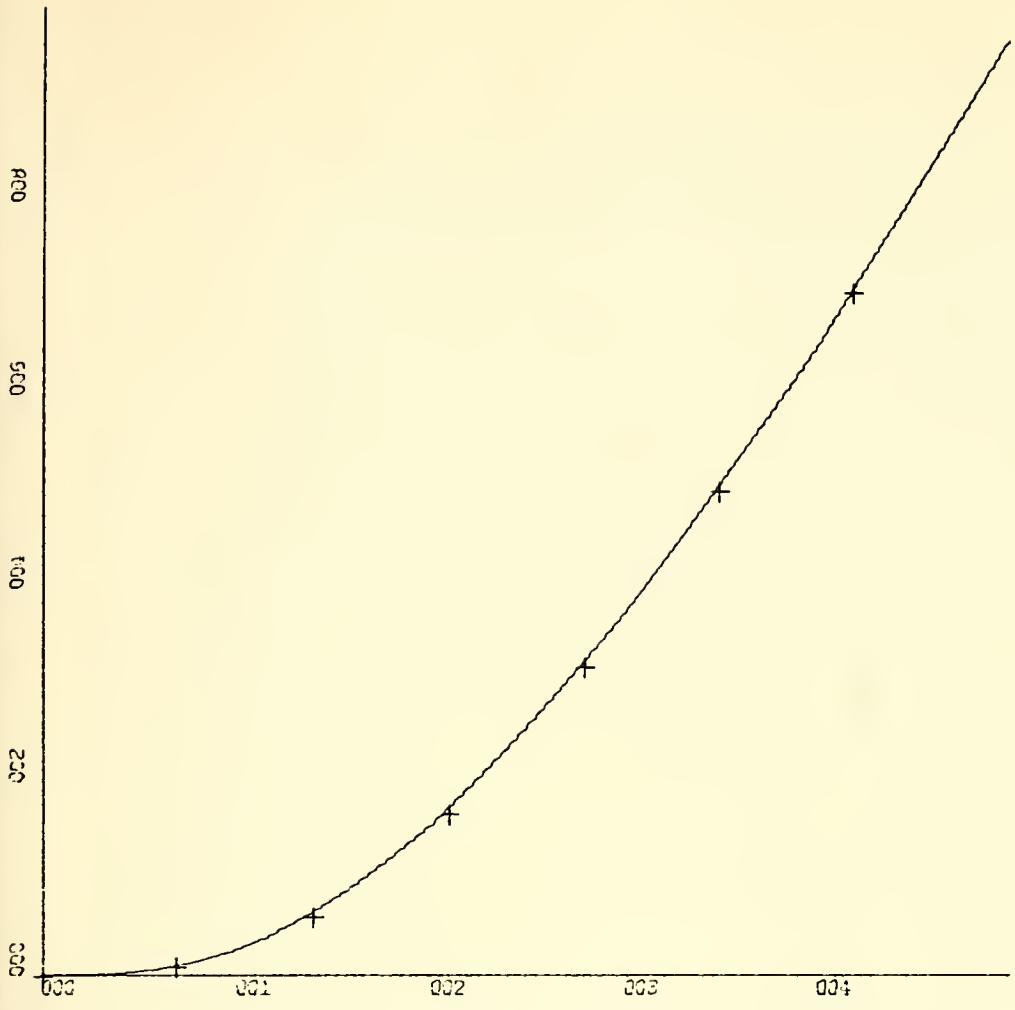
FIGURE 93



FOR DELTA T = 0.6 TFIN = 5.0

T	Q(N)
0.0	-0.00060
0.6	0.05842
1.2	0.41309
1.8	1.17315
2.4	2.29728
3.0	3.69936
3.6	5.30783
4.2	7.07854

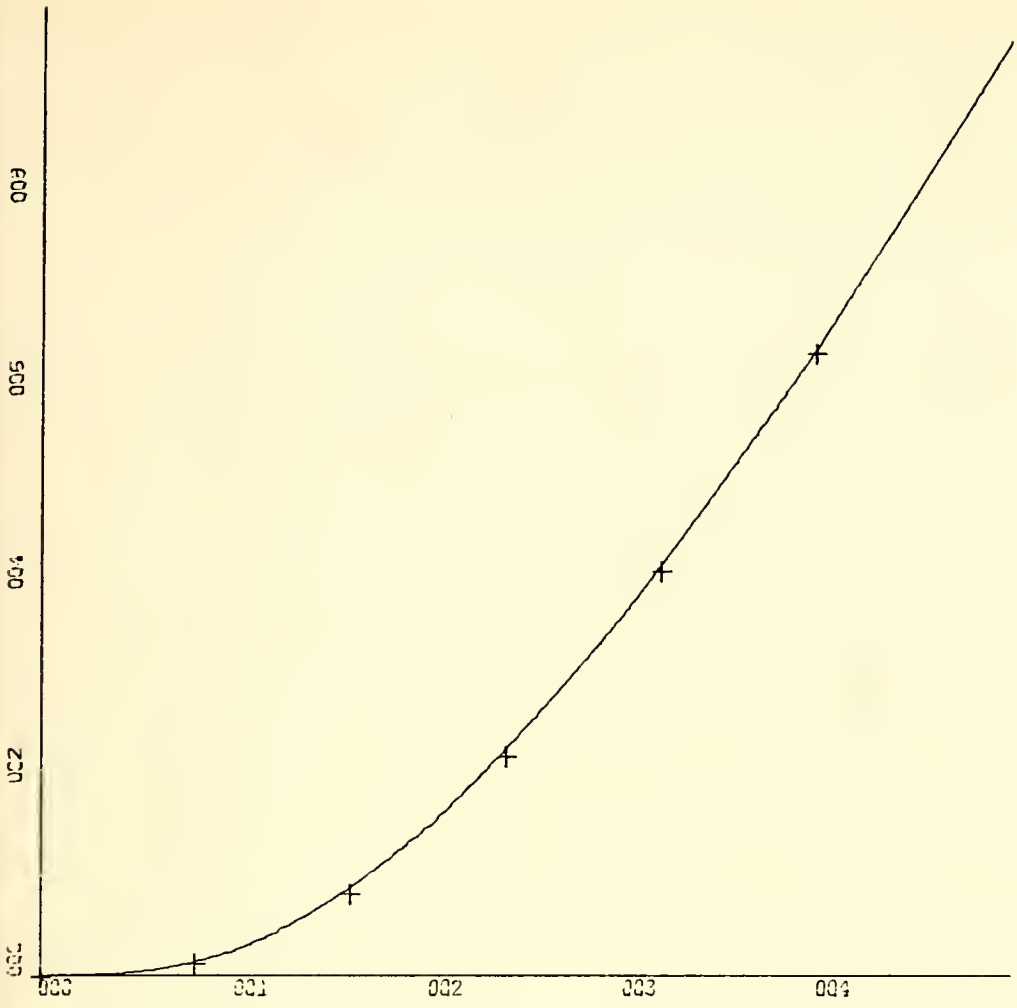
FIGURE 94



FOR DELTA T = 0.7 TFIN = 5.0

T	Q(N)
0.0	-0.00095
0.7	0.08844
1.4	0.61018
2.1	1.67849
2.8	3.18952
3.5	5.01324
4.2	7.06634

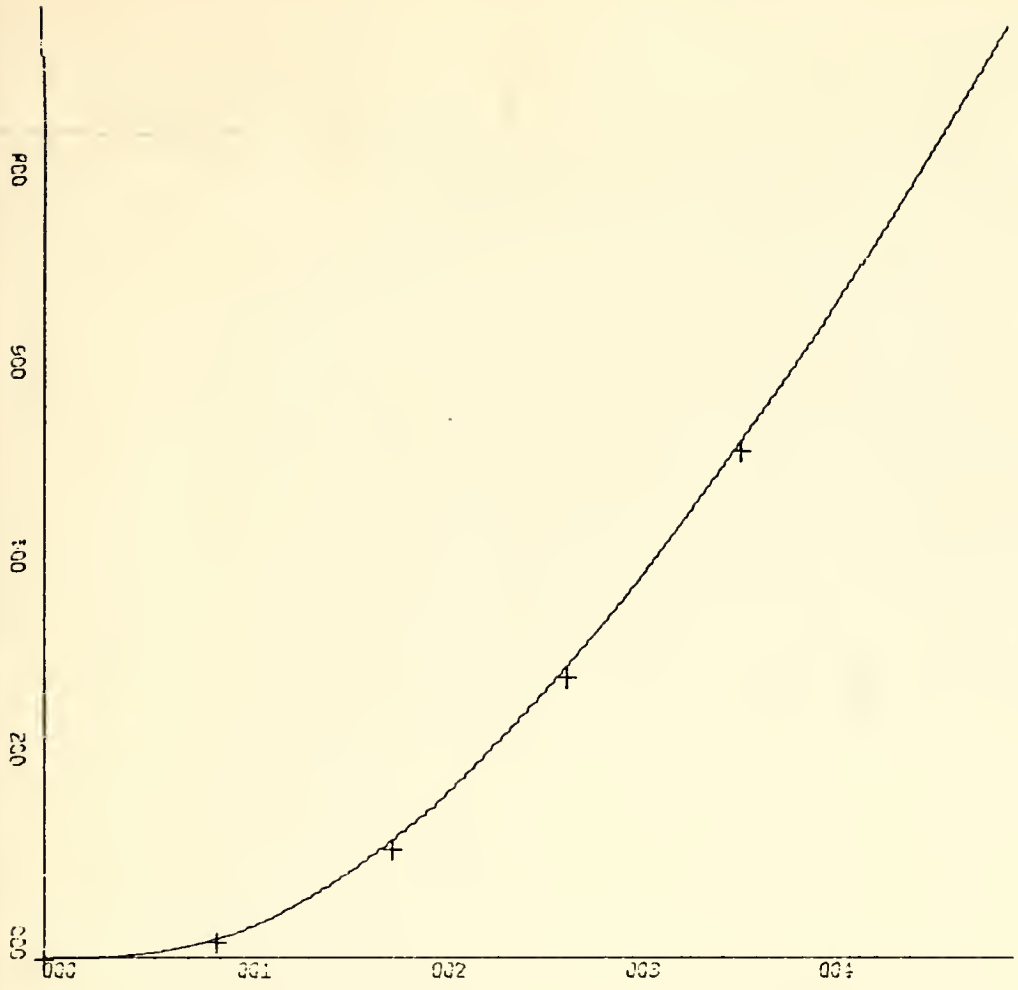
FIGURE 95



FOR DELTA T = 0.8 TFIN = 5.0

T	Q(N)
0.0	-0.00142
0.8	0.12570
1.6	0.84689
2.4	2.26036
3.2	4.18196
4.0	6.44458

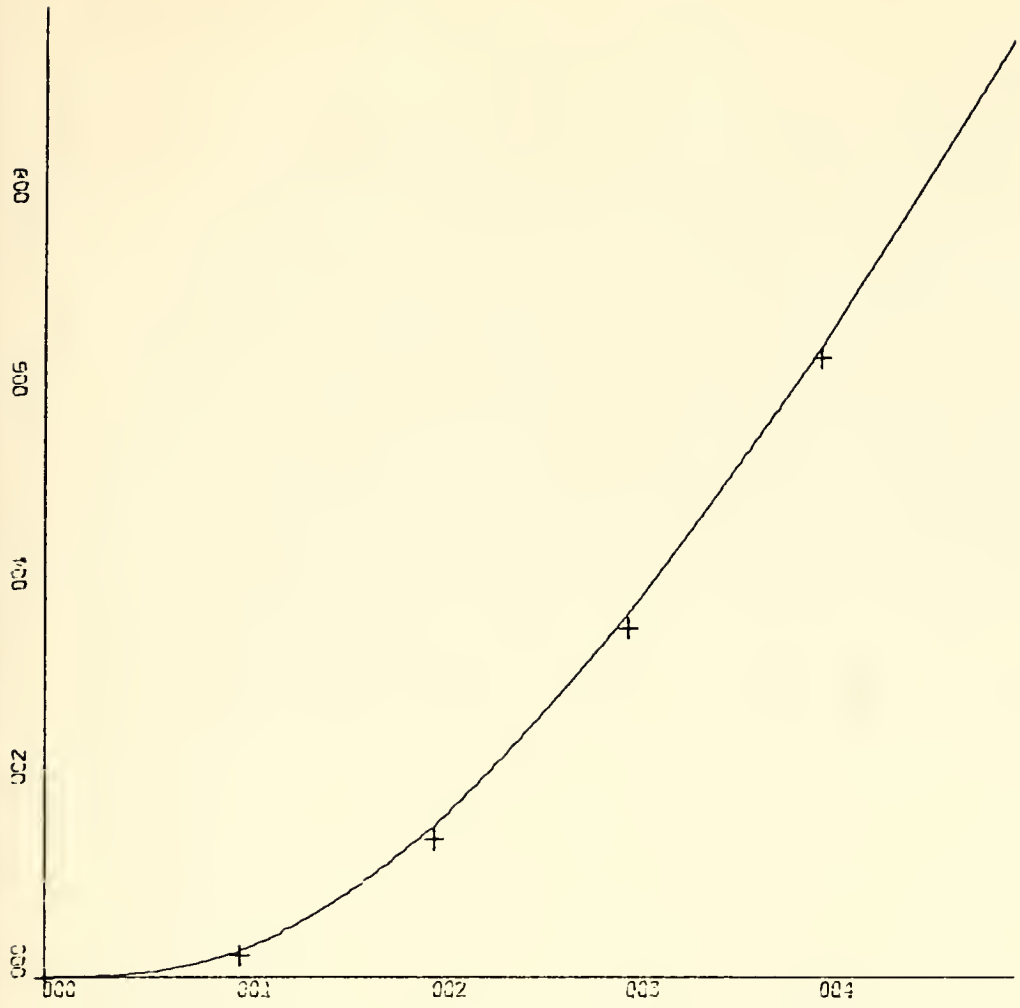
FIGURE 96



FOR DELTA T = 0.9 TFIN = 5.0

T	Q(N)
0.0	-0.00202
0.9	0.17030
1.8	1.12150
2.7	2.90981
3.6	5.25900

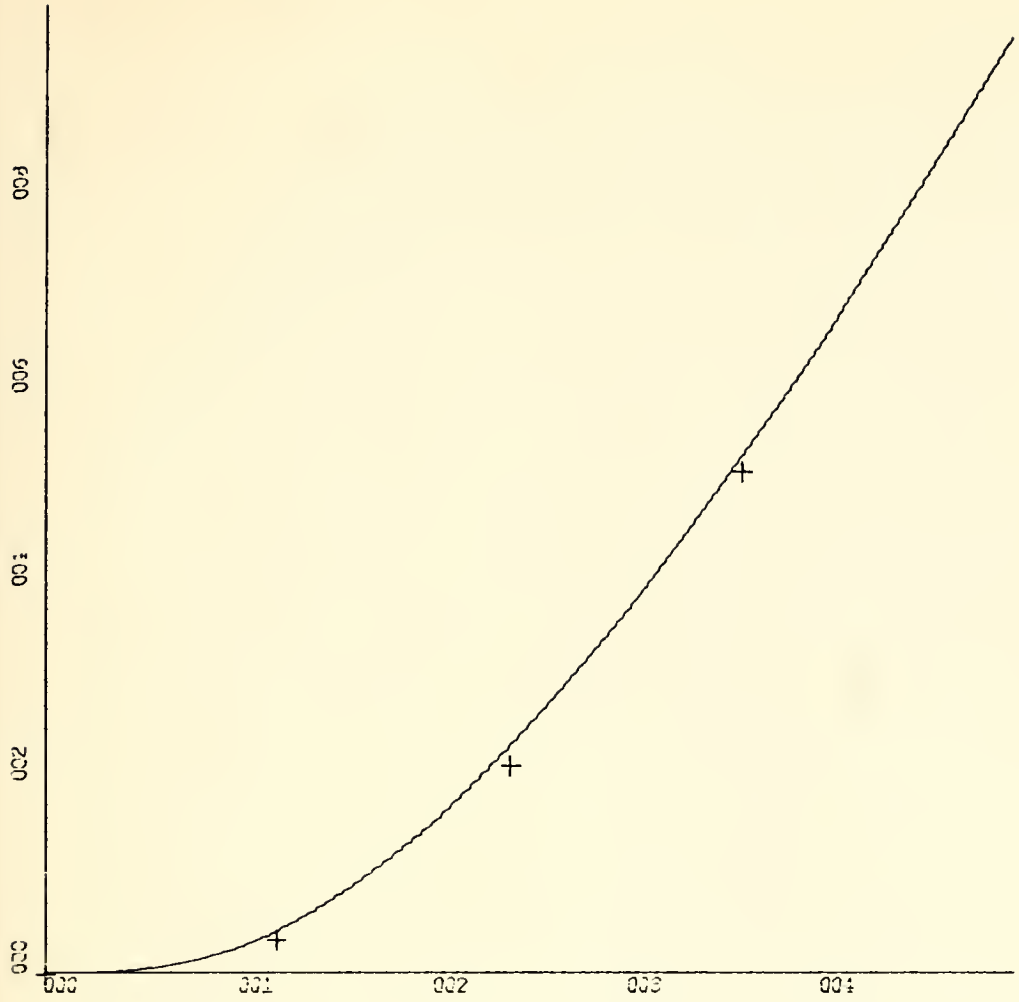
FIGURE 97



FOR DELTA T = 1.0 TFIN = 5.0

T	Q(N)
0.0	-0.00278
1.0	0.22222
2.0	1.43192
3.0	3.61877
4.0	6.40907

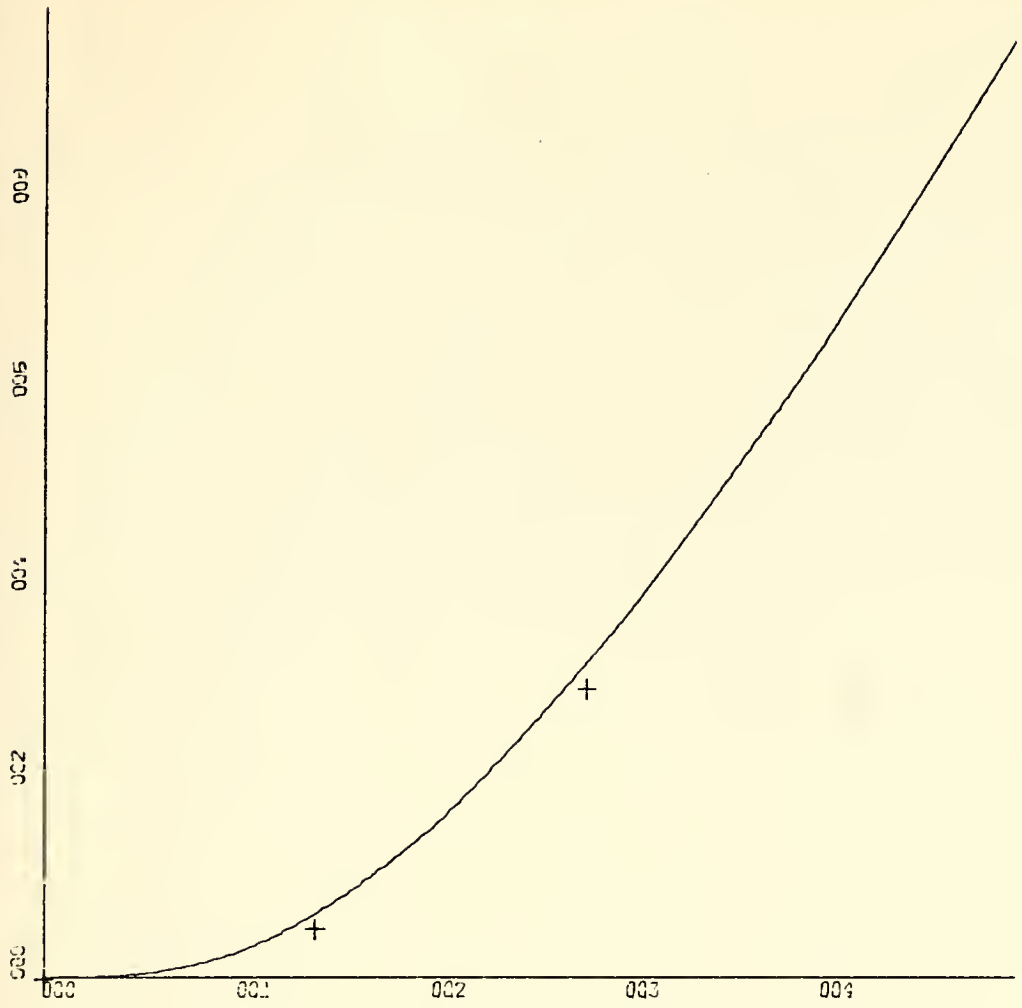
FIGURE 98



FOR DELTA T = 1.2 TFIN = 5.0

T	Q(N)
0.0	-0.00480
1.2	0.34756
2.4	2.15138
3.6	5.18870

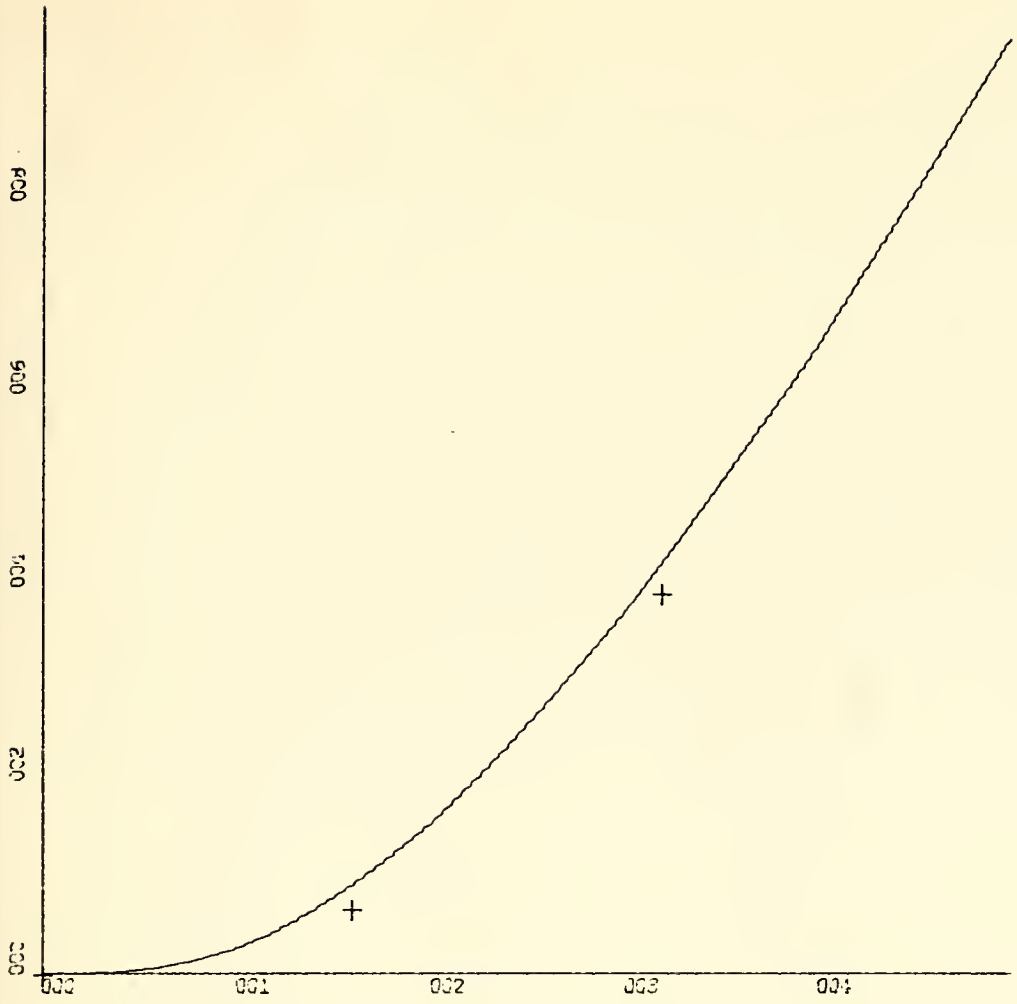
FIGURE 99



FOR DELTA T = 1.4 TFIN = 5.0

T	Q(N)
0.0	-0.00762
1.4	0.50030
2.8	2.98831

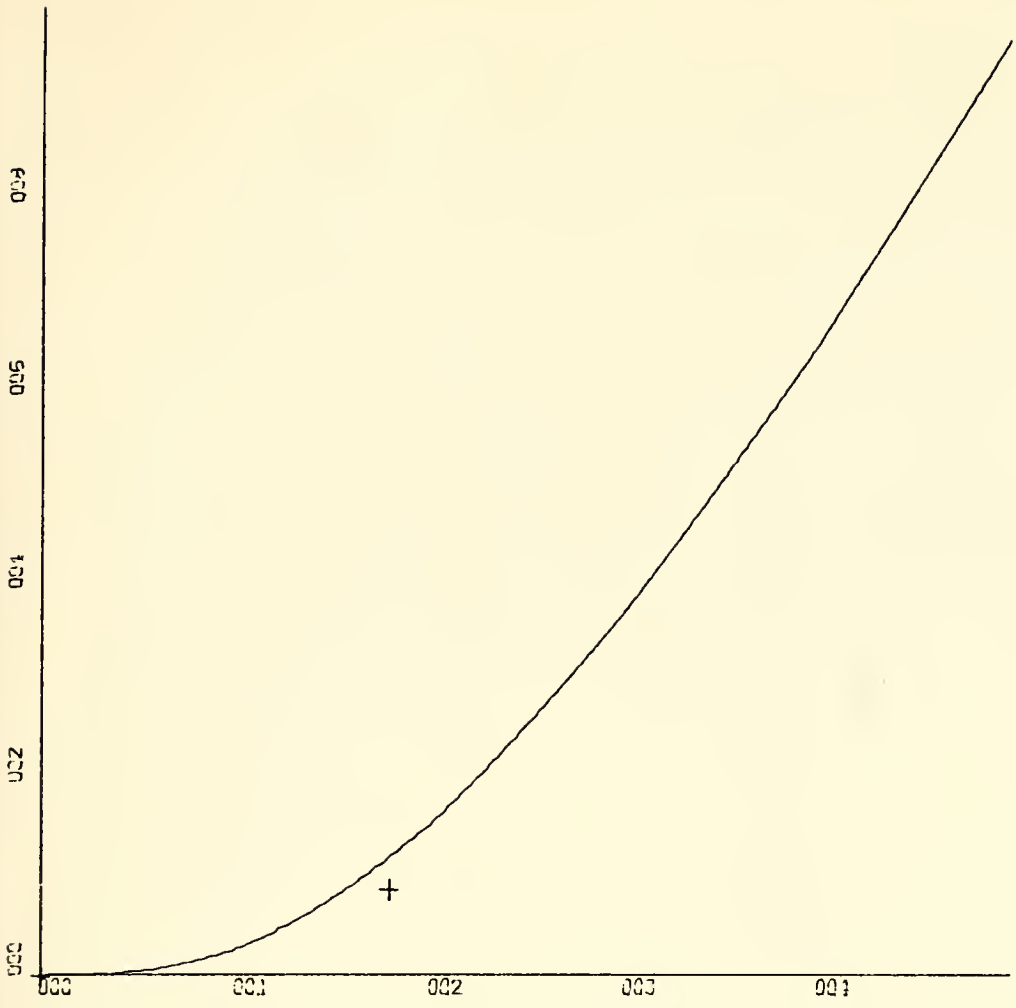
FIGURE 100



FOR DELTA T = 1.6 TFIN = 5.0

T	Q(N)
0.0	-0.01138
1.6	0.67862
3.2	3.92797

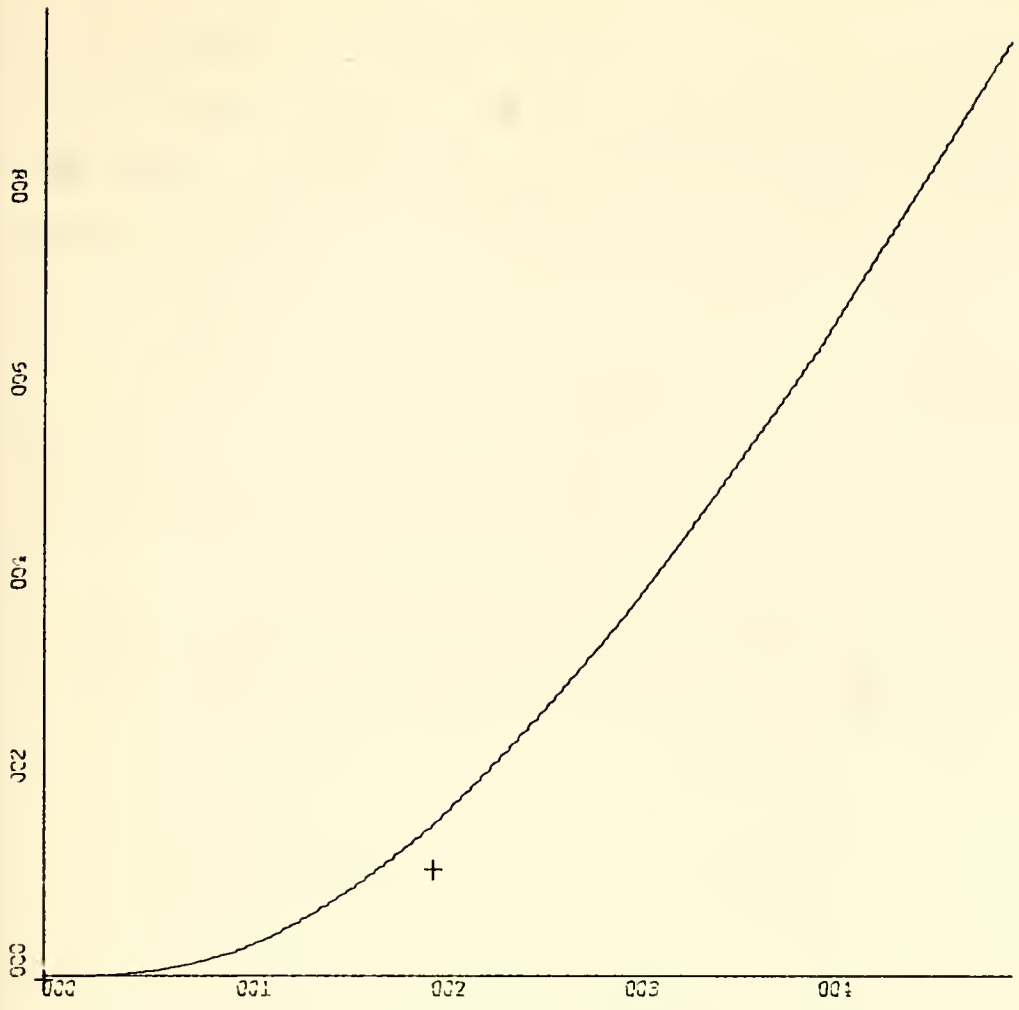
FIGURE 101



FOR DELTA T = 1.8 TFIN = 5.0

T	Q(N)
0.0	-0.01620
1.8	0.88064

FIGURE 102



FOR DELTA T = 2.0 TFIN = 5.0

T	Q(N)
0.0	-0.02222
2.0	1.10457

FIGURE 103

1. Case 8 — $\dot{Y} + tY = t^2$

Same calculations as in case 7 with exception for the substitution on the value of P during the division steps, of Equation (47), repeated here, which now has the value P = t.

The algorithm for computer solution (Program 8, page 144), the output for several time iterations and the respective graphs follow (Figs. 104 to 117)

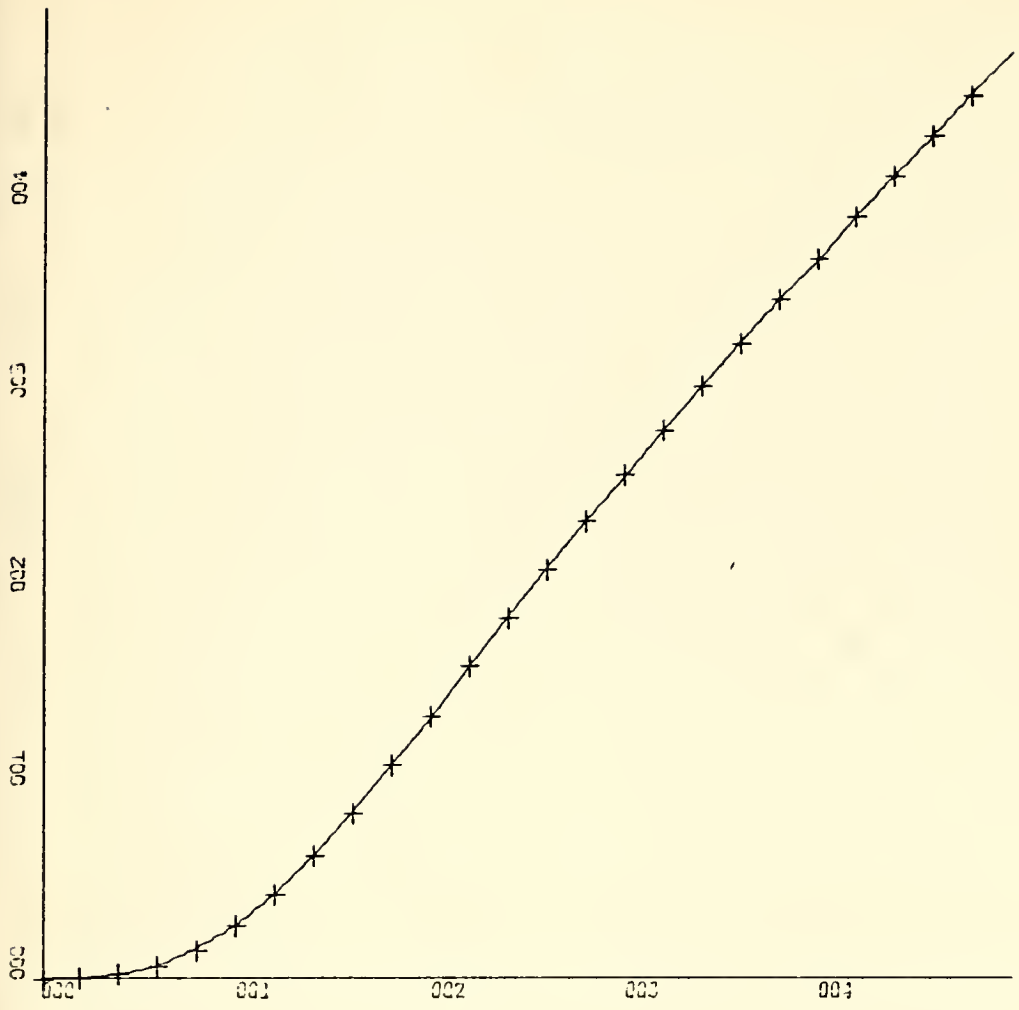
$$y_A^*(z) = \frac{-T^3+124T^3)z^{-1}+(474T^3z^{-2}+124T^3z^{-3}-T^3z^{-4}}{360+180PT-(1440+360PT)z^{-1}+2160z^{-2}-(1440-360PT)z^{-3}+\ldots \ldots+(360-180PT)z^{-4}}$$

FOR DELTA T = 0.2 TFIN = 5.0

T	Q(N)	PR
0.0	-0.00002	0.0
0.2	0.00261	0.00264
0.4	0.02041	0.02066
0.6	0.06629	0.06707
0.8	0.14905	0.15068
1.0	0.27254	0.27522
1.2	0.43567	0.43942
1.4	0.63339	0.63801
1.6	0.85809	0.86328
1.8	1.10124	1.10660
2.0	1.35473	1.35990
2.2	1.61184	1.61660
2.4	1.86765	1.87190
2.6	2.11908	2.12280
2.8	2.36463	2.36780
3.0	2.60393	2.60670
3.2	2.83735	2.83980
3.4	3.06563	3.06780
3.6	3.28958	3.29160
3.8	3.51003	3.51190
4.0	3.72765	3.72940
4.2	3.94301	3.94460
4.4	4.15655	4.15810
4.6	4.36861	4.37000
4.8	4.57945	4.58080

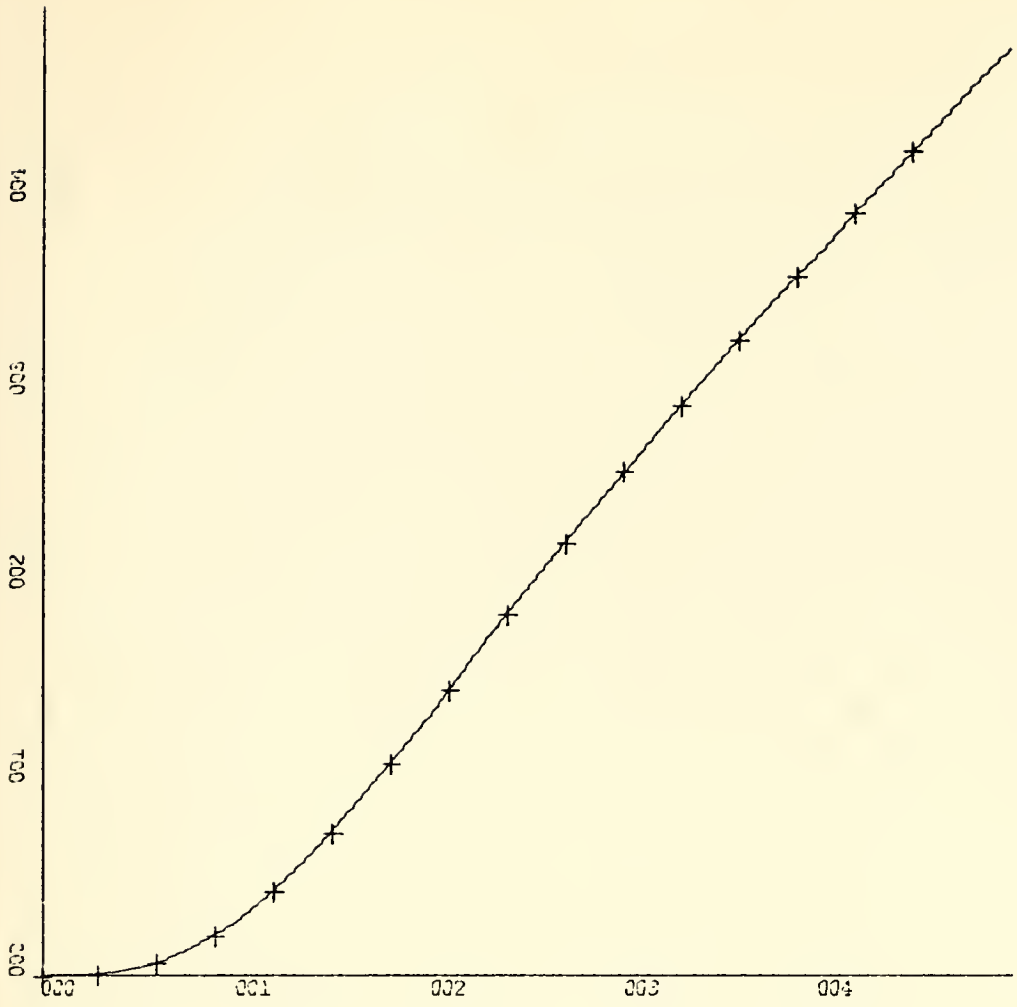
```
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
C  Q(T) = T**2  &  P(T) =T
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
      INTEGER *4ITB(12)/12*0/
      REAL *4RTB(28)/28*0.0/
      DIMENSION A(100), Q(100), X(100), XX(100), PR(100)
      ITB(3) = 5
      ITB(4) = 5
      WRITE (6,8)
C
      DO 1 I=1,26
      READ (5,5) XX(I),PR(I)
      WRITE (6,5) XX(I),PR(I)
    1 CONTINUE
C
    2 READ (5,5,END=4) TD,TFIN
      WRITE (6,6) TD,TFIN
      WRITE (6,7)
      M = TFIN/TD
      T3 = TD**3
      A(1) = -T3
      A(2) = 124.0*T3
      A(3) = 474.0*T3
      A(4) = 124.0*T3
      A(5) = -T3
C
      DO 3 N=1,M
      T = (N-1)*TD
      P = T
      PT = P*TD
      F1 = 360.0+180.0*PT
      F2 = -(1440.0+360.0*PT)
      F3 = 2160.0
      F4 = -(1440.0-360.0*PT)
      F5 = 360.0-180.0*PT
C
      Q(N) = A(N)/F1
C
      A(N+1) = A(N+1)-(Q(N)*F2)
      A(N+2) = A(N+2)-(Q(N)*F3)
      A(N+3) = A(N+3)-(Q(N)*F4)
      A(N+4) = A(N+4)-(Q(N)*F5)
      A(N+5) = 0.0
      WRITE (6,5) T,Q(N)
      X(N) = T
    3 CONTINUE
C
      ITB(1) = 1
      ITB(2) = 0
      ITB(12) = 1
      CALL DRAWP (26,XX,PR,ITB,RTB)
      ITB(1) = 3
      ITB(2) = 2
      CALL DRAWP (M,X,Q,ITB,RTB)
      GO TO 2
    4 STOP
C
    5 FORMAT (2F10.5)
    6 FORMAT ('1','FOR DELTA T=',F6.2,4X,'TFIN=',F4.1)
    7 FORMAT ('  T=',8X,'  Q(N)=')
    8 FORMAT ('   XX=',7X,'   PR=')
      END
```

PROGRAM 8



x-scale = 1.0 units/inch
y-scale = 1.0 units/inch
FOR DELTA T = 0.2

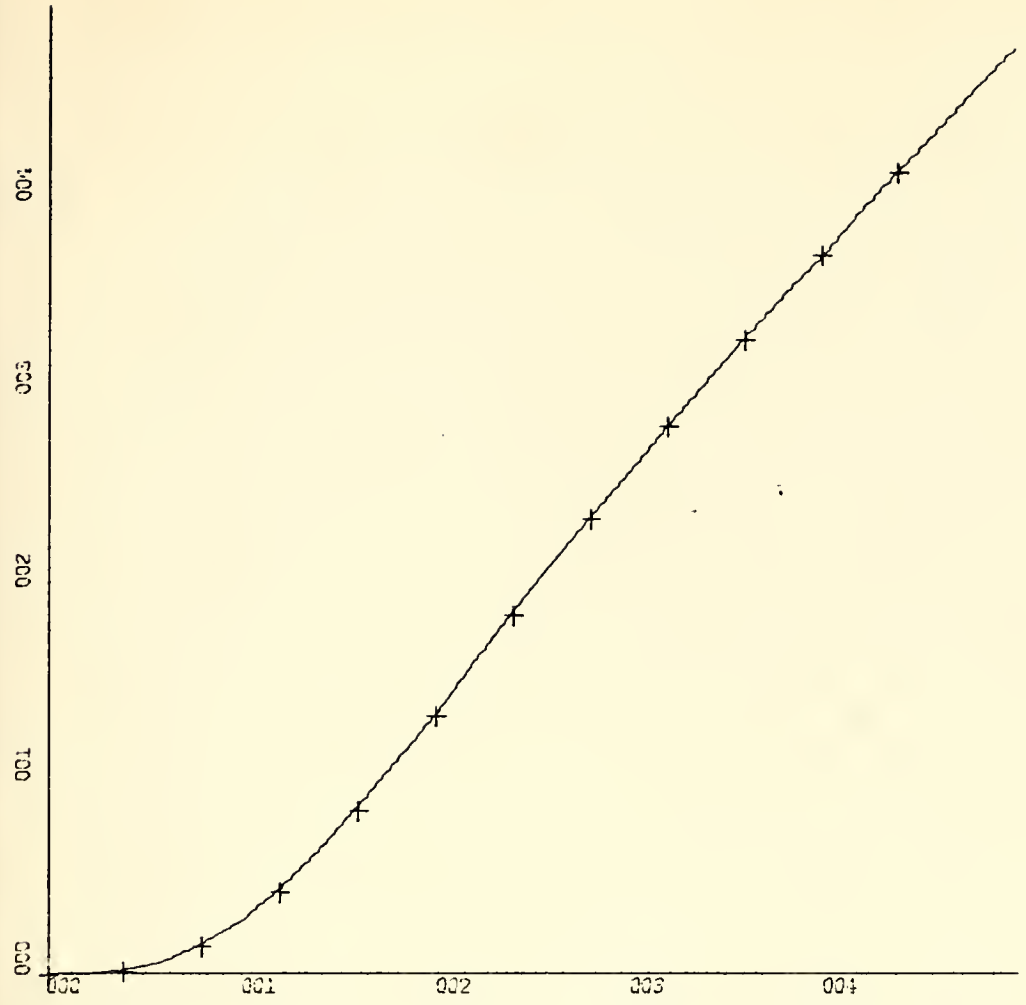
FIGURE 104



FOR DELTA T = 0.3 TFIN = 5.0

T	Q(N)
0.0	-0.00007
0.3	0.00861
0.6	0.06534
0.9	0.20305
1.2	0.43105
1.5	0.73670
1.8	1.09444
2.1	1.47676
2.4	1.86218
2.7	2.23828
3.0	2.60051
3.3	2.94932
3.6	3.28723
3.9	3.61706
4.2	3.94109
4.5	4.26094

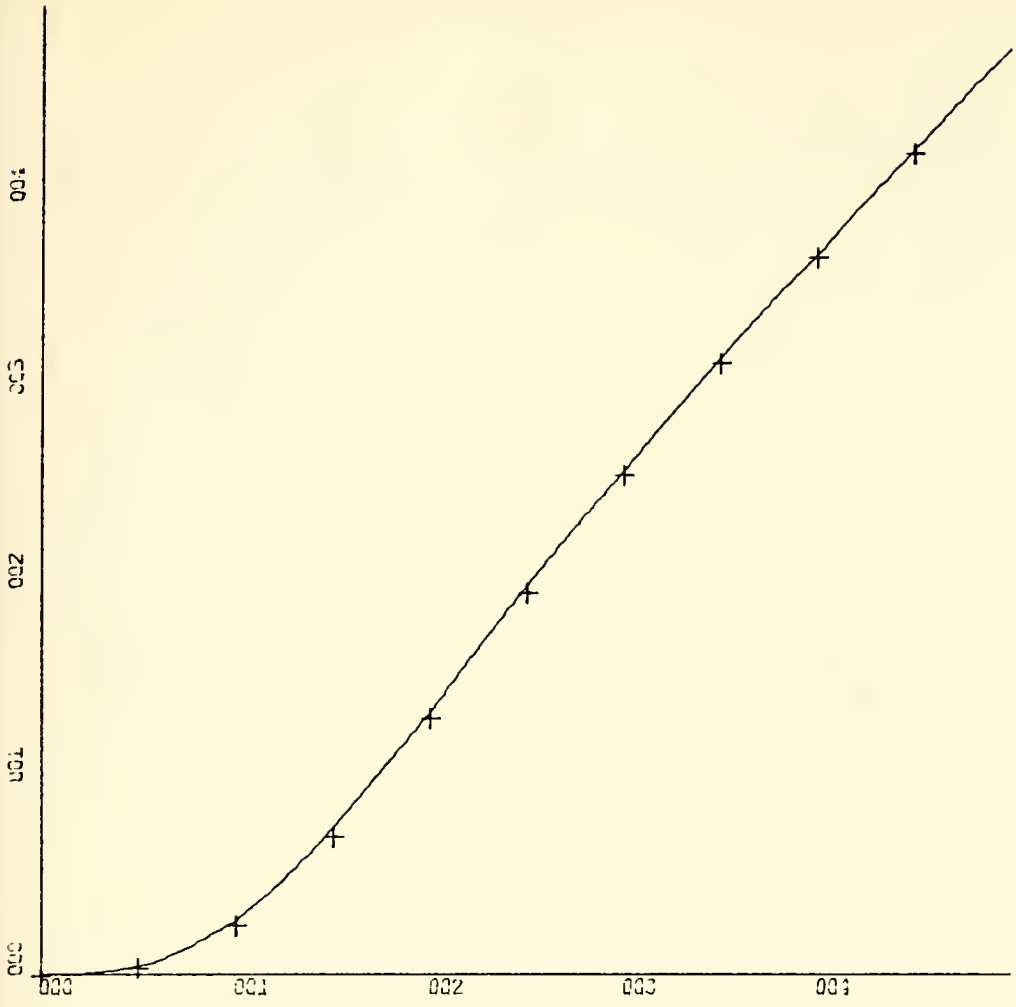
FIGURE 105



FOR DELTA T = 0.4 TFIN = 5.0

T	Q(N)
0.0	-0.00018
0.4	0.01975
0.8	0.14440
1.2	0.42470
1.6	0.84250
2.0	1.33874
2.4	1.85444
2.8	2.35490
3.2	2.83016
3.6	3.28380
4.0	3.72261
4.4	4.15198

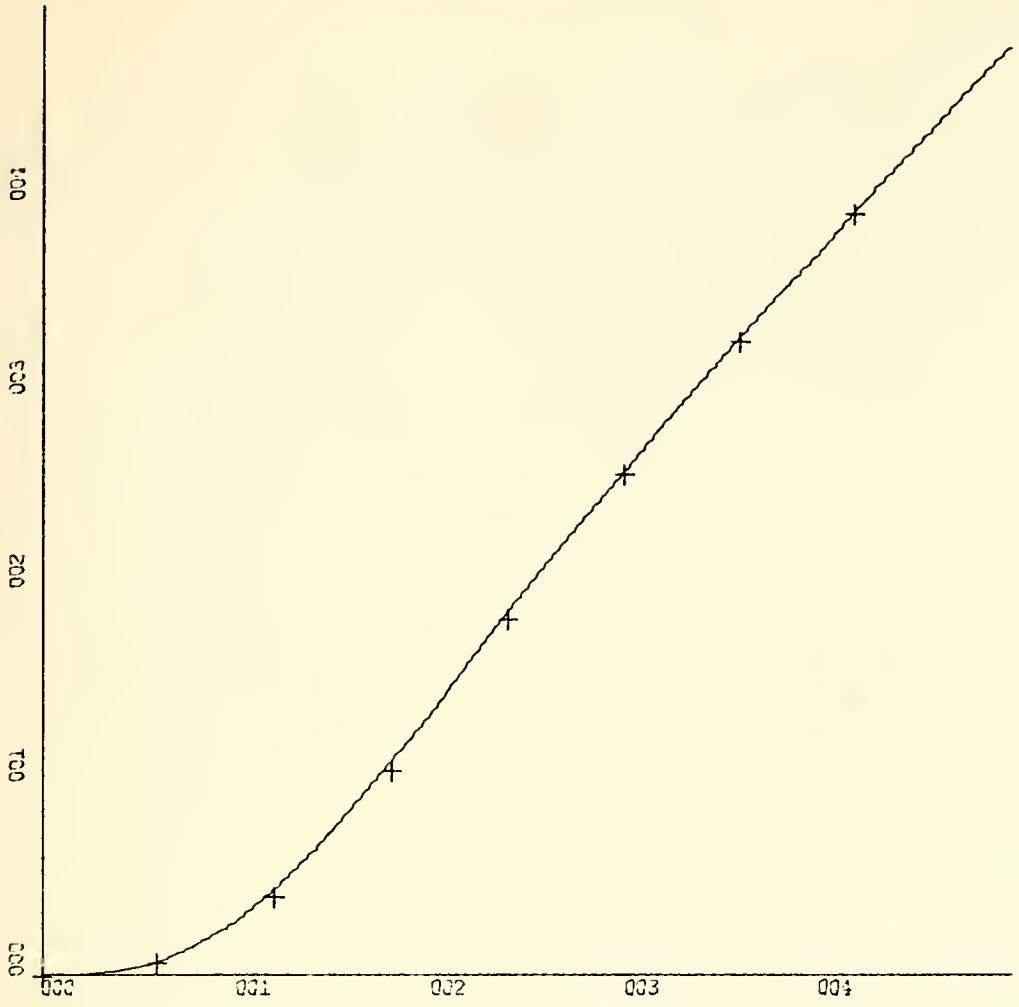
FIGURE 106



FOR DELTA T = 0.5 TFIN = 5.0

T	Q(N)
0.0	-0.00035
0.5	0.03704
1.0	0.25926
1.5	0.71717
2.0	1.32660
2.5	1.97228
3.0	2.58929
3.5	3.16744
4.0	3.71877
4.5	4.25487

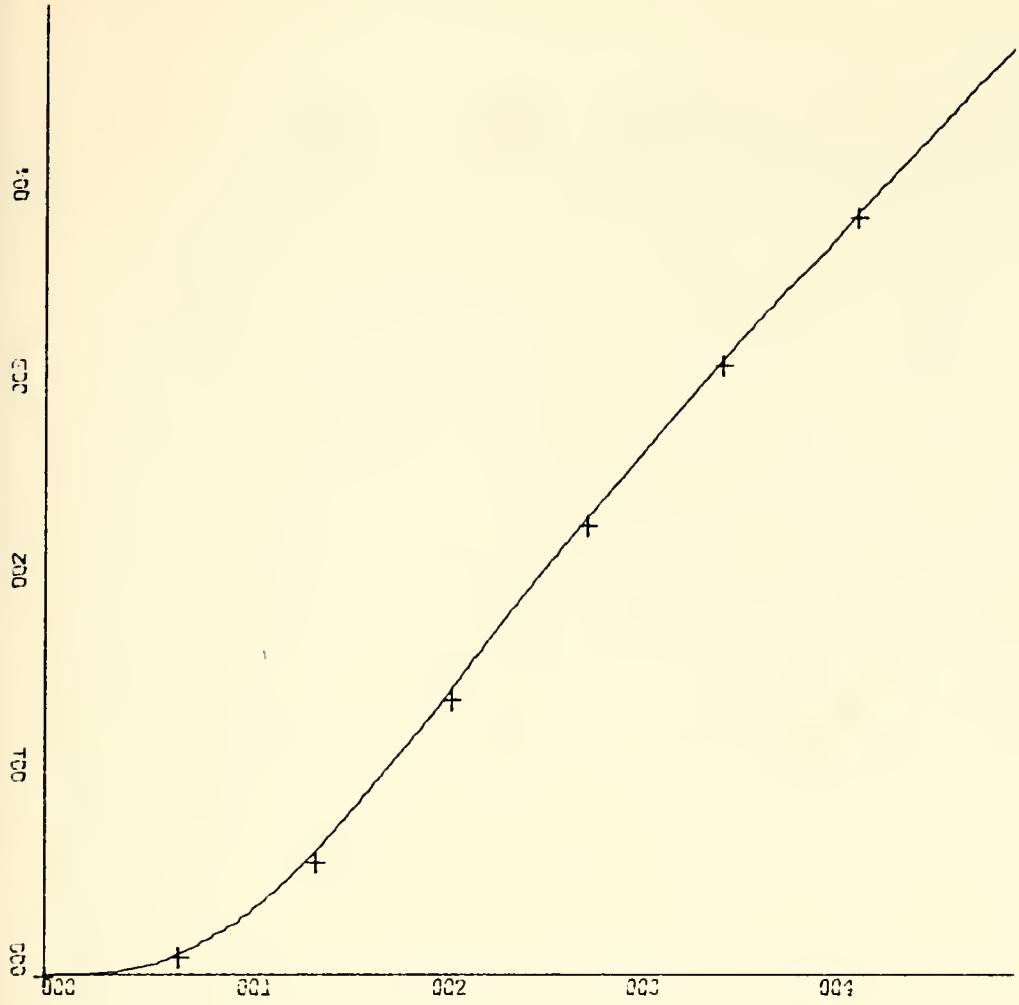
FIGURE 107



FOR DELTA T = 0.6 TFIN = 5.0

T	Q(N)
0.0	-0.00060
0.6	0.06102
1.2	0.40738
1.8	1.05761
2.4	1.83168
3.0	2.58151
3.6	3.27410
4.2	3.93011

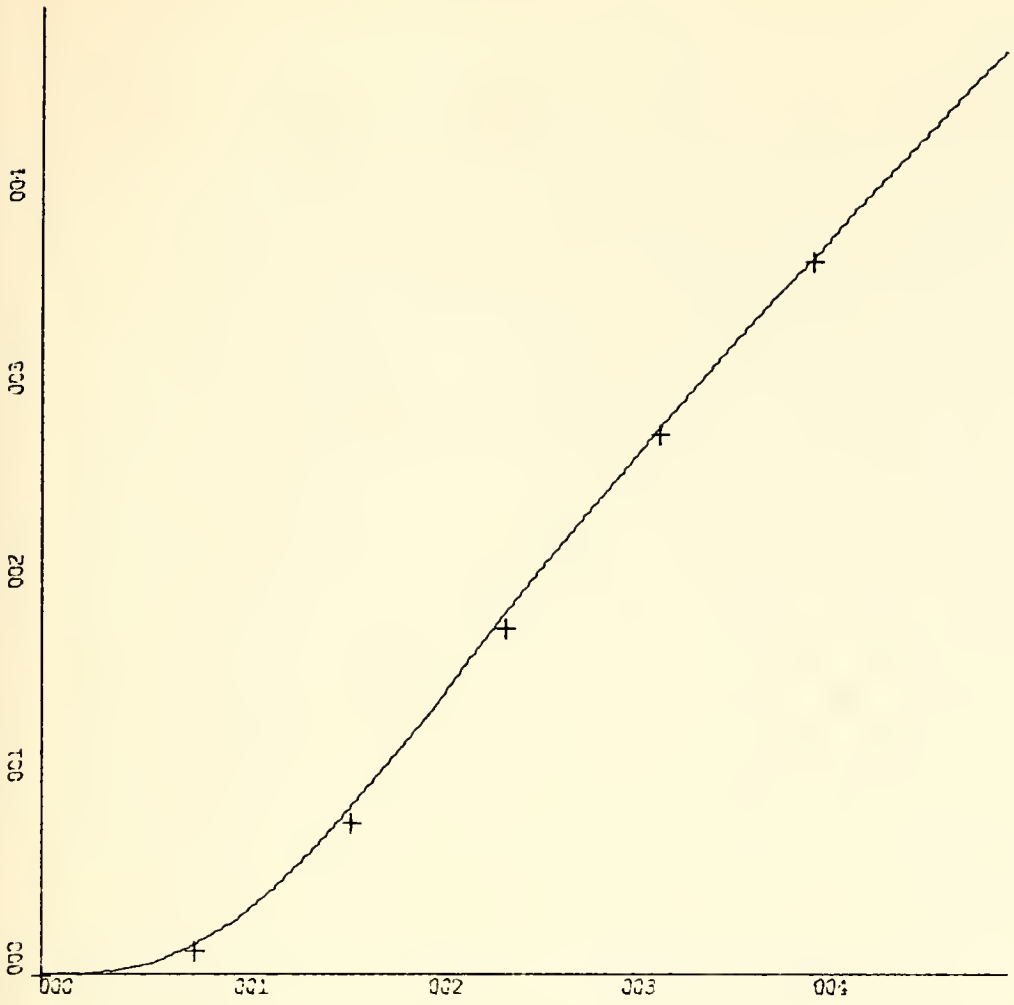
FIGURE 108



FOR DELTA T = 0.7 TFIN = 5.0

T	Q(N)
0.0	-0.00095
0.7	0.09183
1.4	0.58367
2.1	1.42363
2.8	2.32707
3.5	3.15544
4.2	3.92482

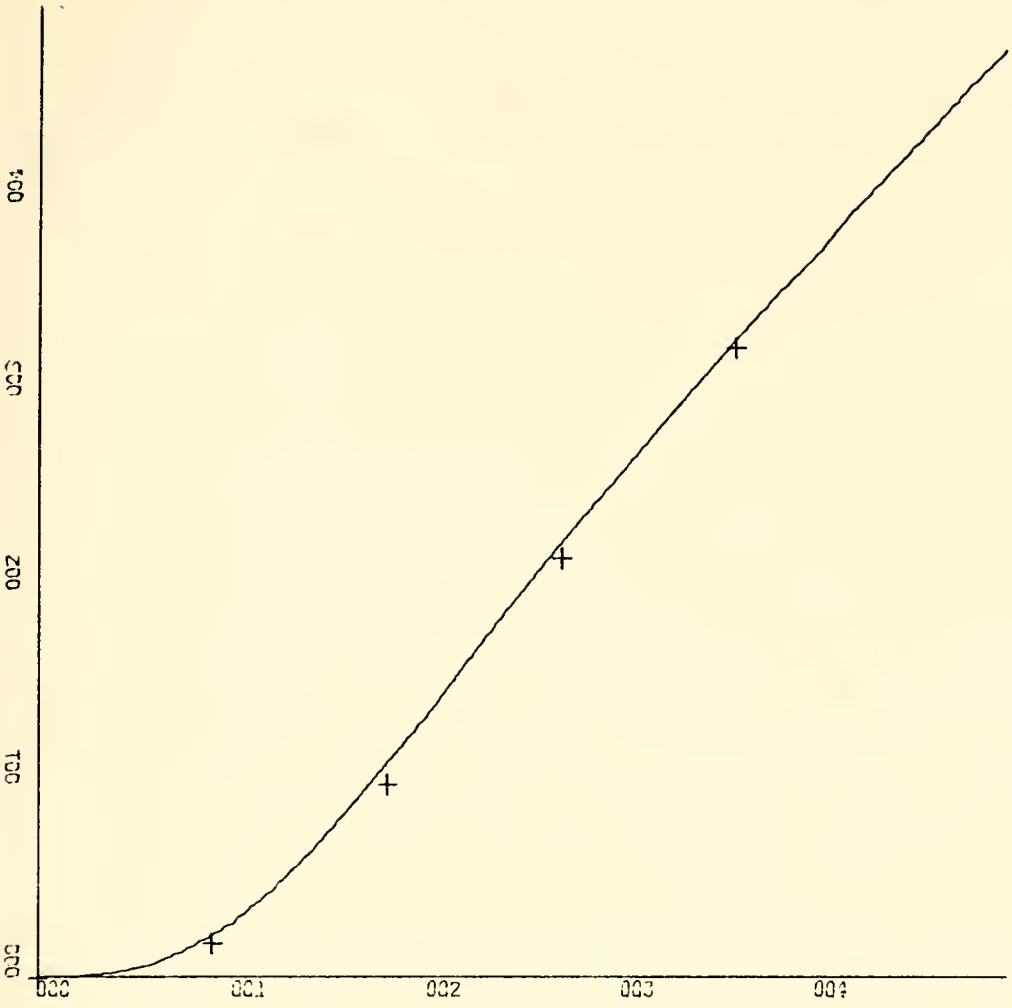
FIGURE 109



FOR DELTA T = 0.8 TFIN = 5.0

T	Q(N)
0.0	-0.00142
0.8	0.12929
1.6	0.78206
2.4	1.79807
3.2	2.80113
4.0	3.70243

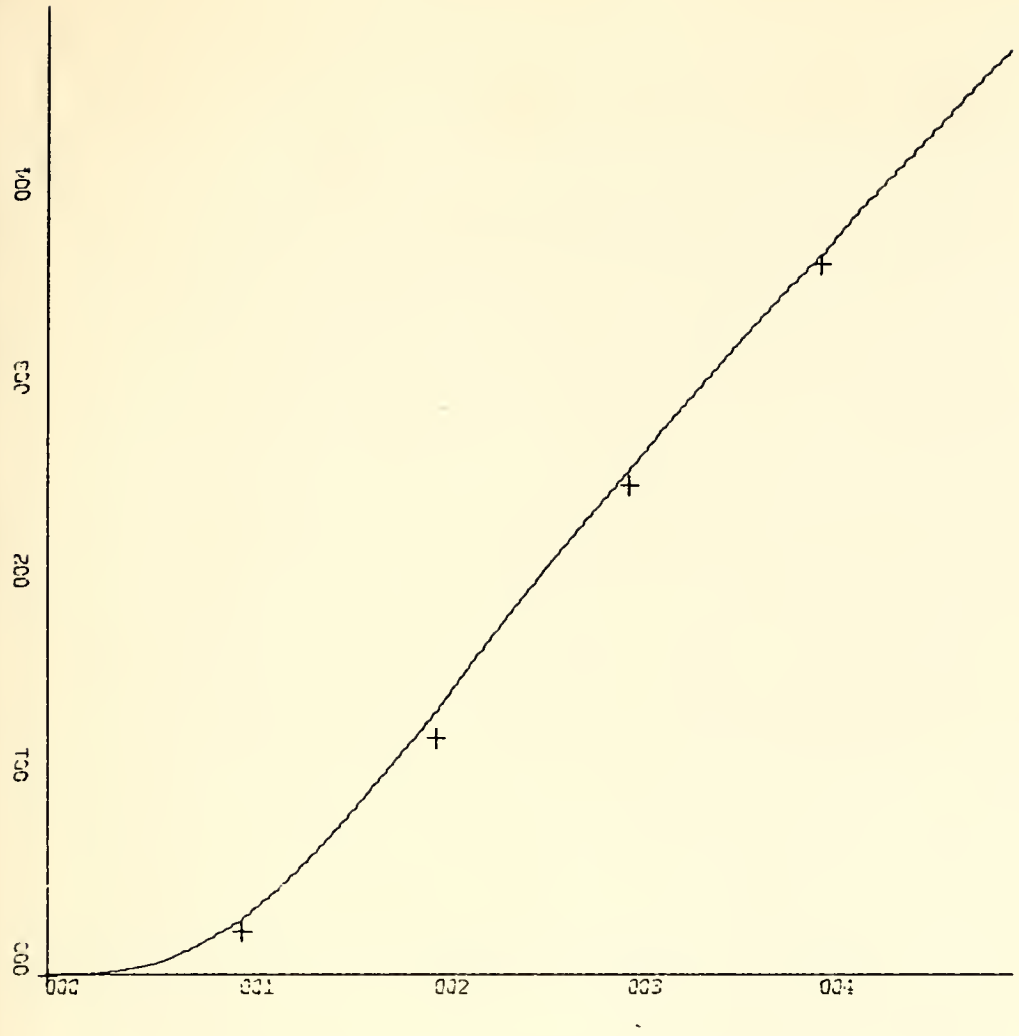
FIGURE 110



FOR DELTA T = 0.9 TFIN = 5.0

T	Q(N)
0.0	=0.00202
0.9	0.17295
1.8	0.99663
2.7	2.16991
3.6	3.25361

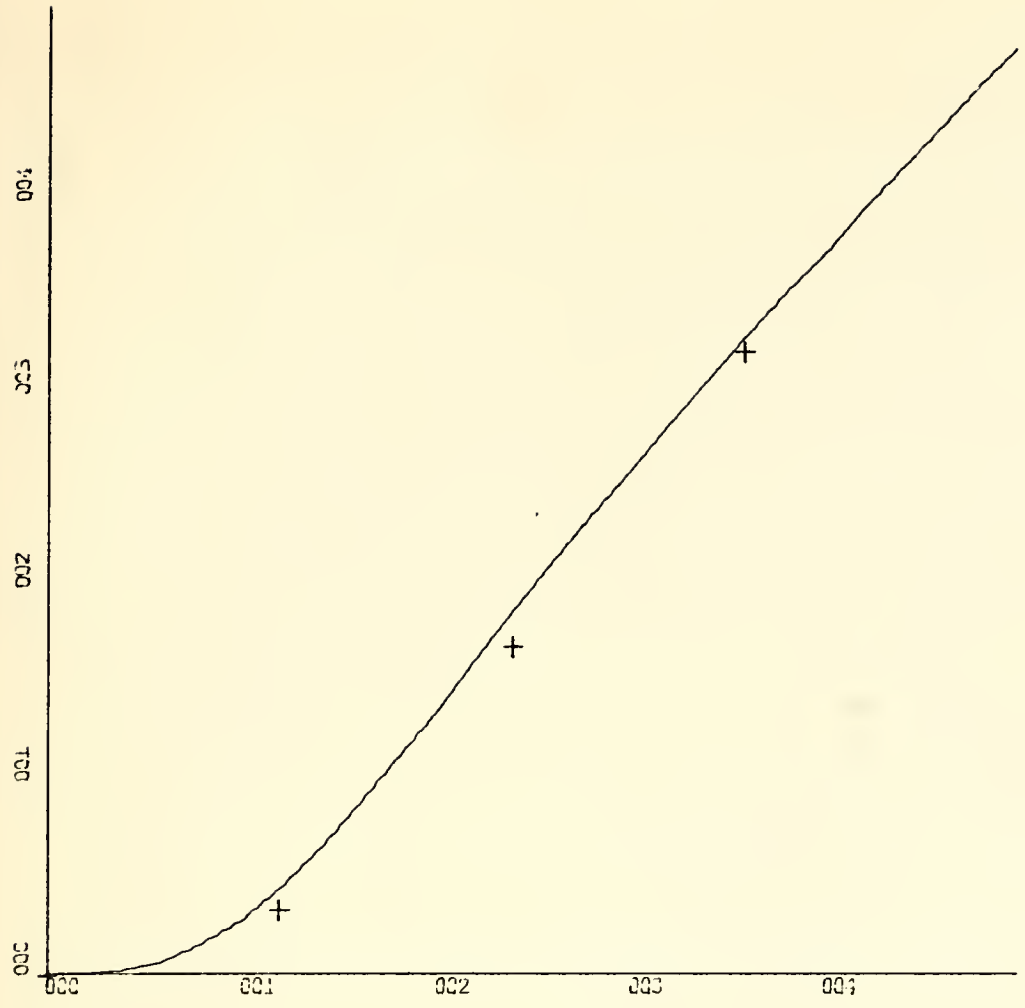
FIGURE 111



FOR DELTA T = 1.0 TFIN = 5.0

T	Q(N)
0.0	-0.00278
1.0	0.22222
2.0	1.22222
3.0	2.53333
4.0	3.68888

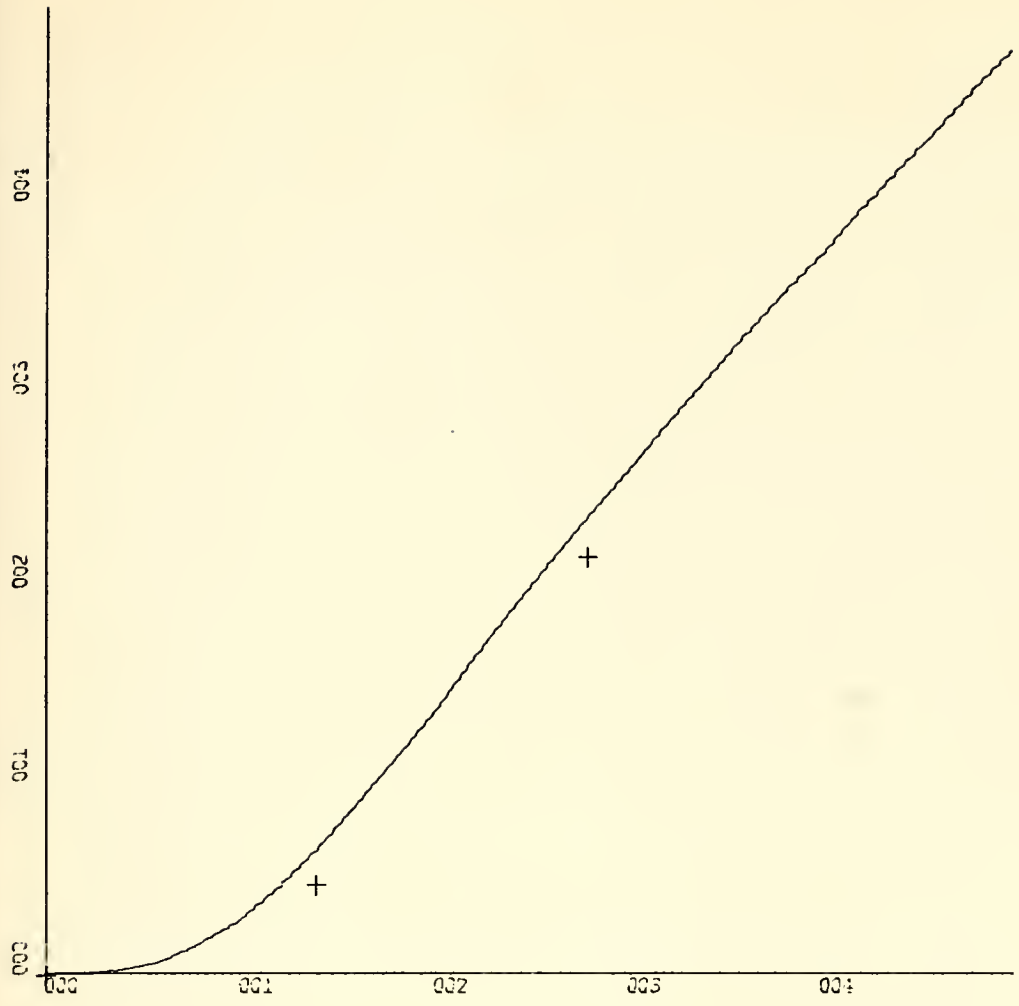
FIGURE 112



FOR DELTA T = 1.2 TFIN = 5.0

T	Q(N)
0.0	-0.00480
1.2	0.33488
2.4	1.69089
3.6	3.22784

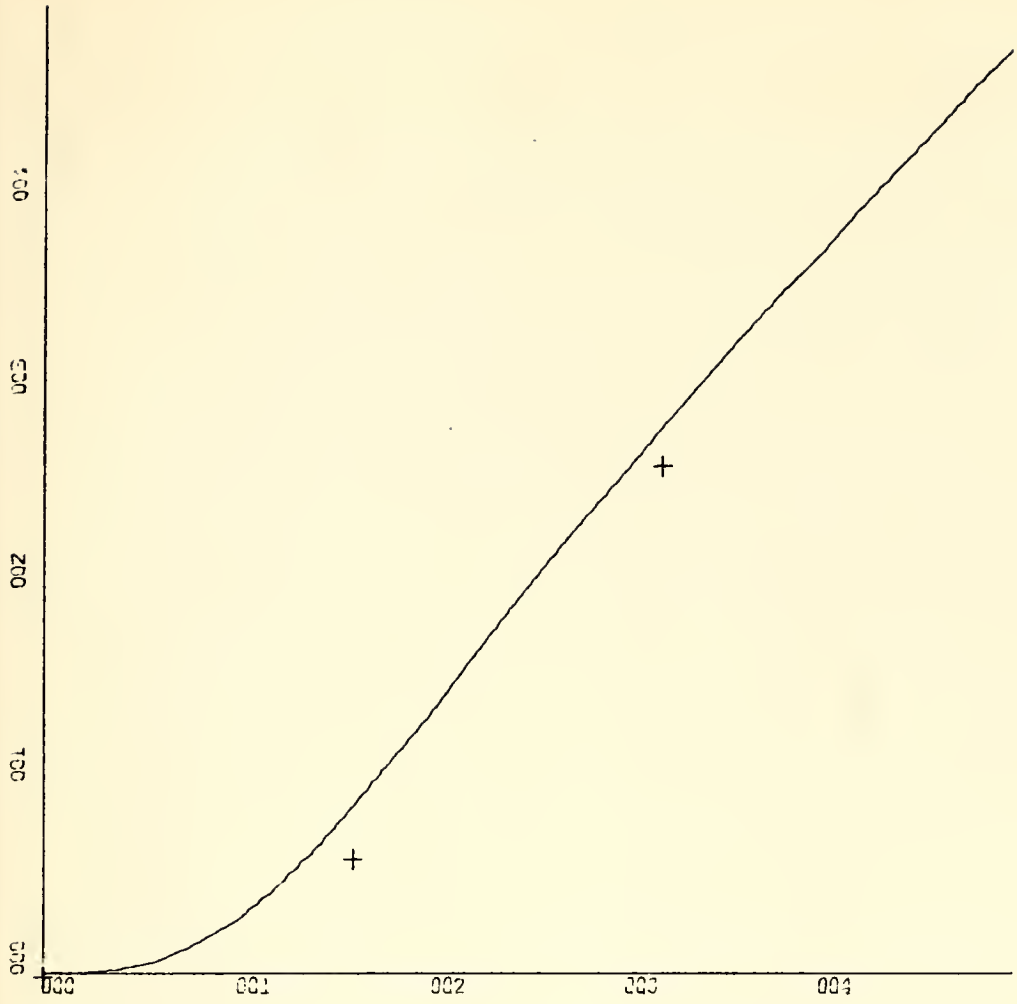
FIGURE 113



FOR DELTA T = 1.4 TFIN = 5.0

T	Q(N)
0.0	-0.00762
1.4	0.46195
2.8	2.16618

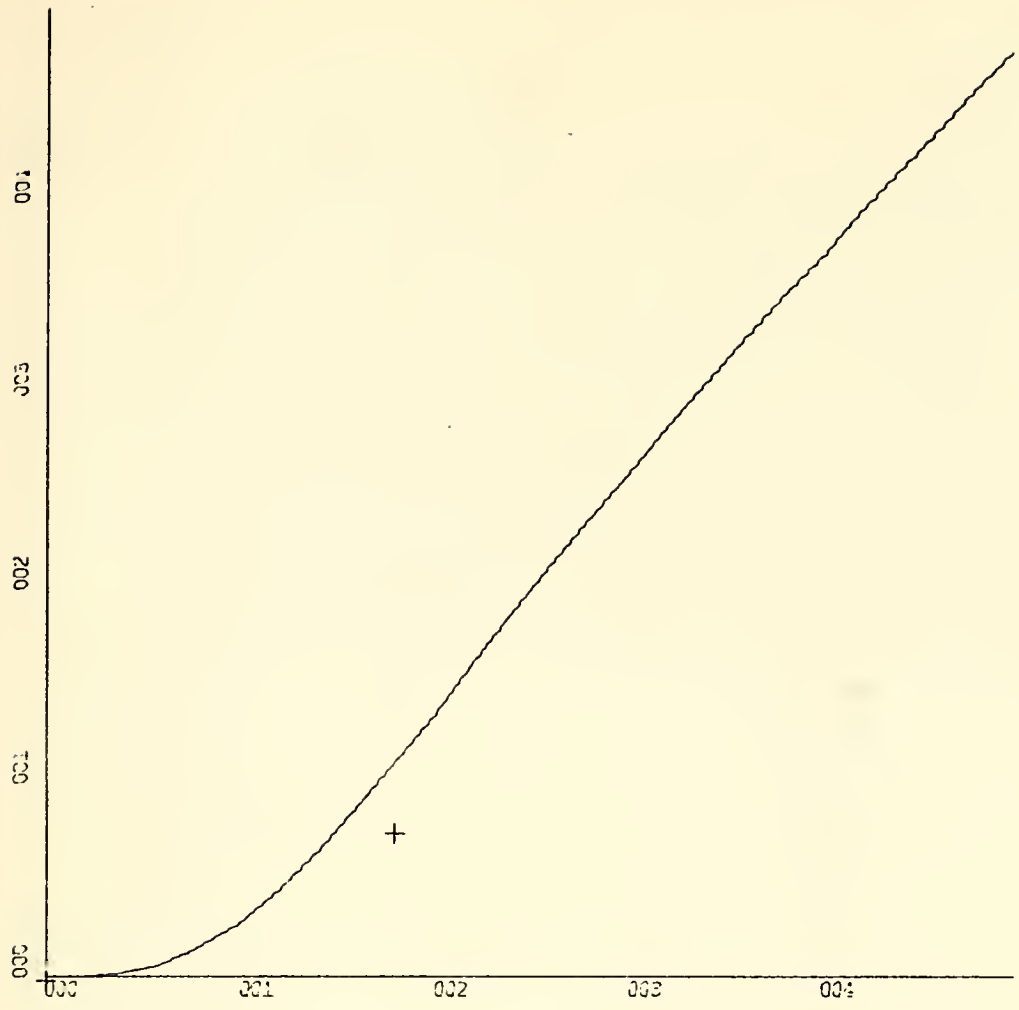
FIGURE 114



FOR DELTA T = 1.6 TFIN = 5.0

T	Q(N)
0.0	-0.01138
1.6	0.59883
3.2	2.63754

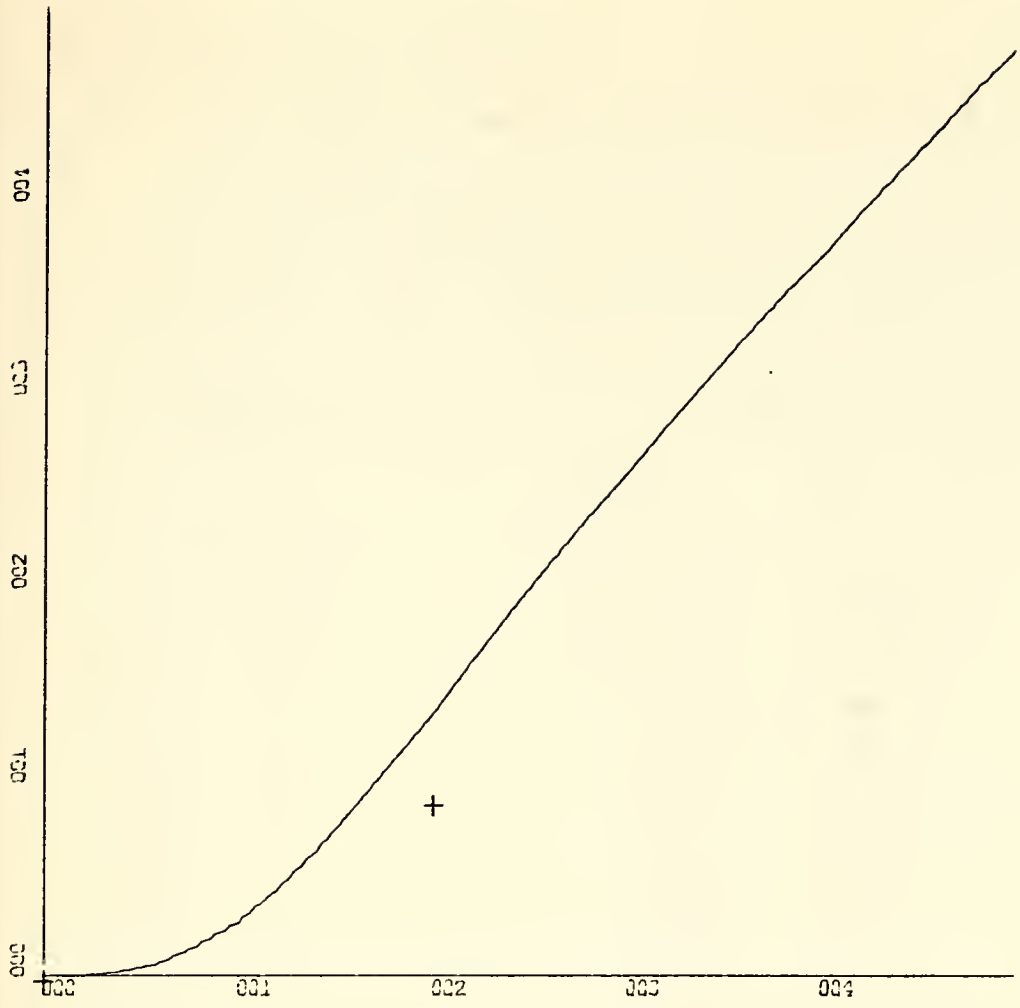
FIGURE 115



FOR DELTA T = 1.8 TFIN = 5.0

T	Q(N)
0.0	-0.01620
1.8	0.74198

FIGURE 116



FOR DELTA T = 2.0 TFIN = 5.0

T	Q(N)
0.0	-0.02222
2.0	0.88889

FIGURE 117

2. Case 9 — $\dot{Y} + t^2Y = t^2$

Same reasoning as in case 8. This time $P = t^2$.

Computer algorithm for solution of this problem is shown in Program 9 (Page 160). Solutions and graphs for several T's follow in Figures 118 to 131.

FOR DELTA T = 0.2 TFIN = 5.0

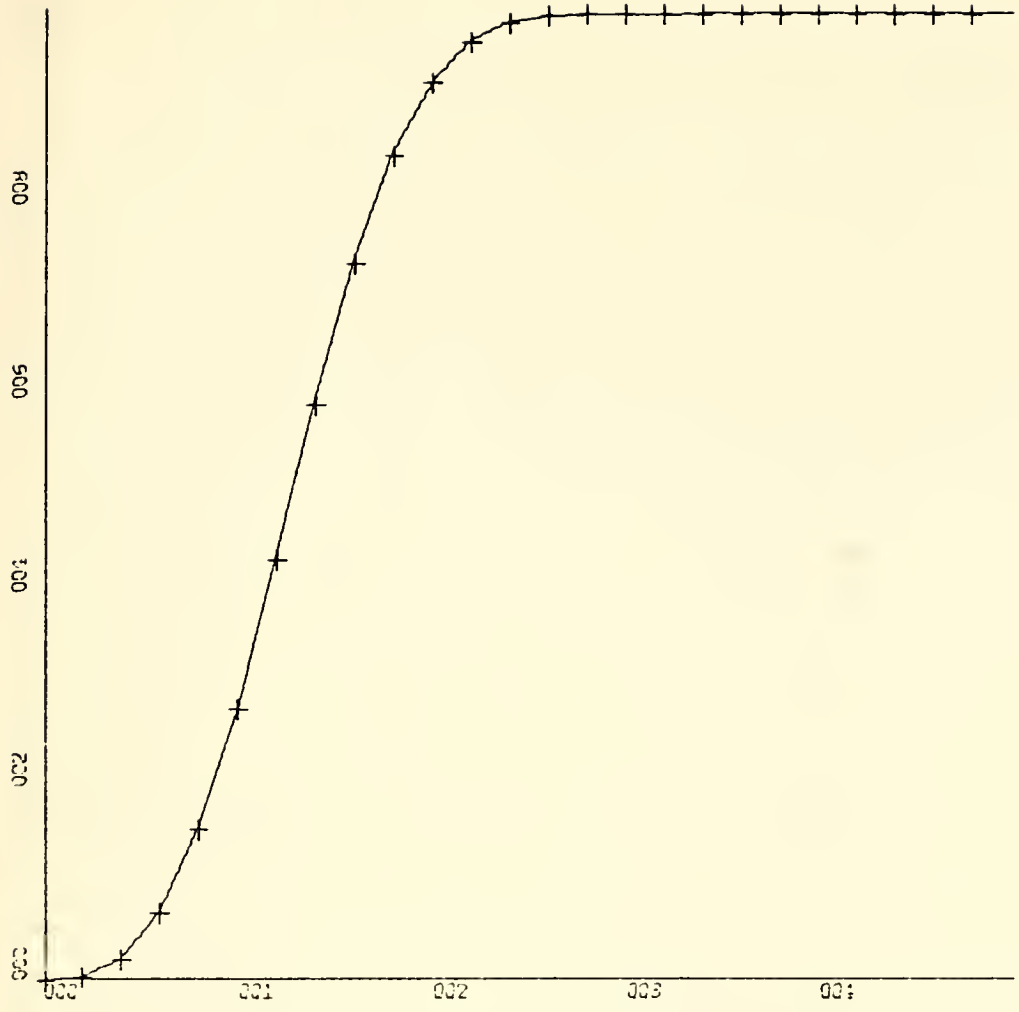
T	Q(N)	PR
0.0	-0.00002	0.0
0.2	0.00266	0.00266
0.4	0.02098	0.02110
0.6	0.06883	0.06947
0.8	0.15509	0.15690
1.0	0.27985	0.28347
1.2	0.43228	0.43785
1.4	0.59256	0.59934
1.6	0.73812	0.74469
1.8	0.85183	0.85686
2.0	0.92749	0.93051
2.2	0.96977	0.97125
2.4	0.98924	0.99002
2.6	0.99645	0.99714
2.8	0.99857	0.99934
3.0	0.99909	0.99988
3.2	0.99924	0.99998
3.4	0.99932	1.00000
3.6	0.99938	1.00000
3.8	0.99943	1.00000
4.0	0.99947	1.00000
4.2	0.99950	1.00000
4.4	0.99952	1.00000
4.6	0.99954	1.00000
4.8	0.99956	1.00000

Equation (47) is repeated for convenience:

$$y_A^*(z) = \frac{-T^3+(124T^3)z^{-1}+(474T^3)z^{-2}+124T^3z^{-3}-T^3z^{-4}}{360+180PT-(1440+360PT)z^{-1}+2160z^{-2}-(1440-360PT)z^{-3}+\ldots \ldots+(360-180PT)z^{-4}}$$

```
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
C   Q(T) = T**2  &  P(T) =T**2
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
C
      INTEGER *4ITB(12)/12*0/
      REAL *4RTB(28)/28*0.0/
      DIMENSION A(100), Q(100), X(100), XX(100), PR(100)
      ITB(3) = 5
      ITB(4) = 5
      WRITE (6,8)
C
      DO 1 I=1,26
      READ (5,5) XX(I),PR(I)
      WRITE (6,5) XX(I),PR(I)
    1 CONTINUE
C
    2 READ (5,5,END=4) TD,TFIN
      WRITE (6,6) TD,TFIN
      WRITE (6,7)
      M = TFIN/TD
      T3 = TD**3
      A(1) = -T3
      A(2) = 124.0*T3
      A(3) = 474.0*T3
      A(4) = 124.0*T3
      A(5) = -T3
C
      DO 3 N=1,M
      T = (N-1)*TD
      P = T**2
      PT = P*TD
      F1 = 360.0+180.0*PT
      F2 = -(1440.0+360.0*PT)
      F3 = 2160.0
      F4 = -(1440.0-360.0*PT)
      F5 = 360.0-180.0*PT
C
      Q(N) = A(N)/F1
C
      A(N+1) = A(N+1)-(Q(N)*F2)
      A(N+2) = A(N+2)-(Q(N)*F3)
      A(N+3) = A(N+3)-(Q(N)*F4)
      A(N+4) = A(N+4)-(Q(N)*F5)
      A(N+5) = 0.0
      WRITE (6,5) T,Q(N)
      X(N) = T
    3 CONTINUE
C
      ITB(1) = 1
      ITB(2) = 0
      ITB(12) = 1
      CALL DRAWP (26,XX,PR,ITB,RTB)
      ITB(1) = 3
      ITB(2) = 2
      CALL DRAWP (M,X,Q,ITB,RTB)
      GO TO 2
    4 STOP
C
    5 FORMAT (2F10.5)
    6 FORMAT ('1','FOR DELTA T=',F6.2,4X,'TFIN=',F4.1)
    7 FORMAT ('  T=',8X,'  Q(N)=')
    8 FORMAT ('   XX=',7X,'   PR=')
      END
```

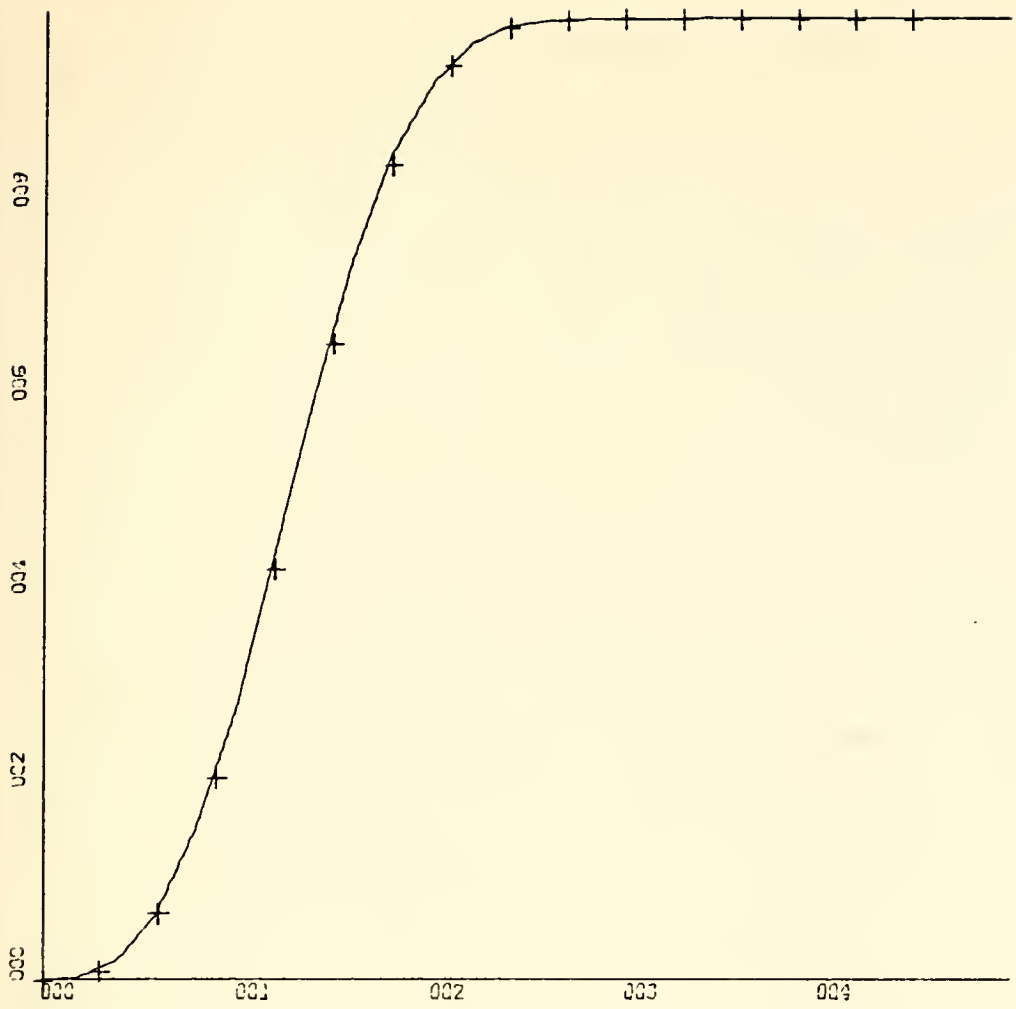
PROGRAM 9



x-scale = 1.0 units/inch
y-scale = 0.2 units/inch

FOR DELTA T = 0.2

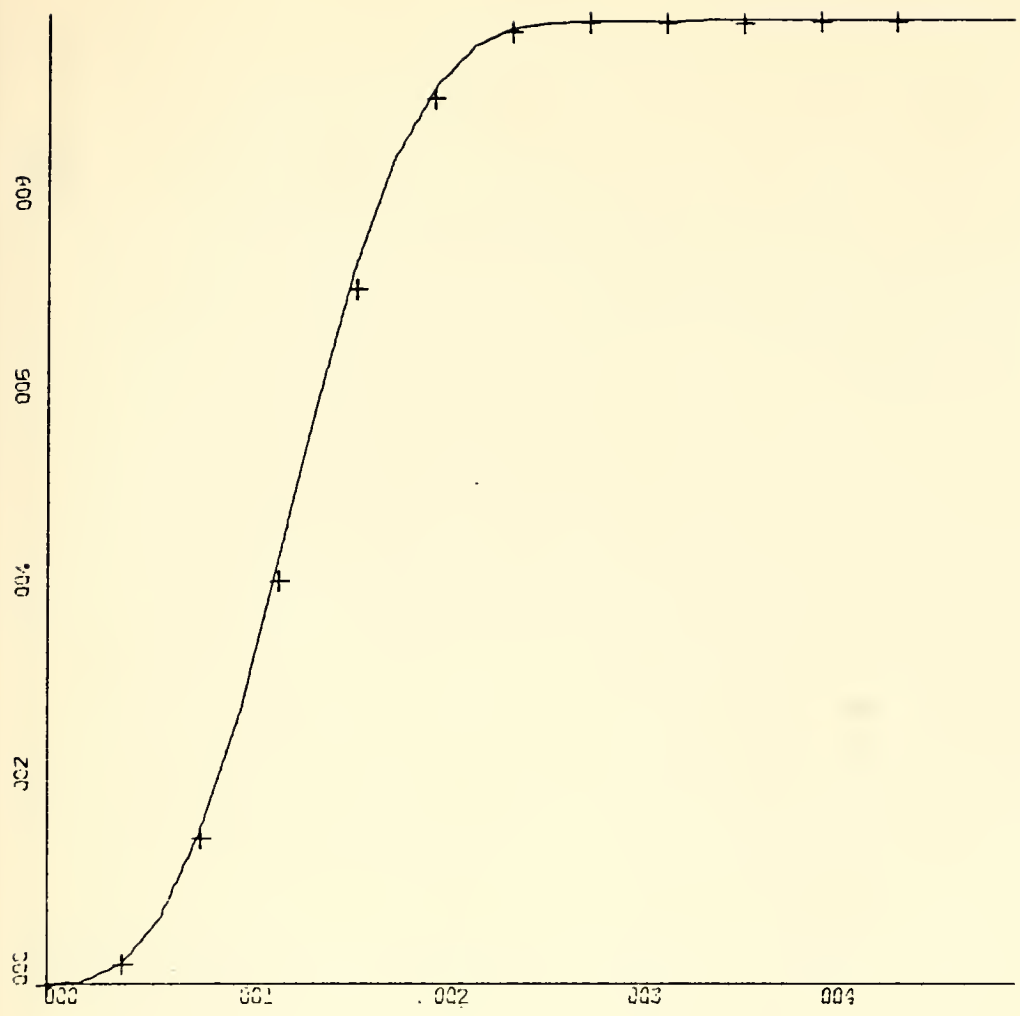
FIGURE 118



FOR DELTA T = 0.3 TFIN = 5.0

T	Q(N)
0.0	-0.00007
0.3	0.00888
0.6	0.06808
0.9	0.20990
1.2	0.42549
1.5	0.65988
1.8	0.84534
2.1	0.94944
2.4	0.98840
2.7	0.99809
3.0	0.99819
3.3	0.99851
3.6	0.99877
3.9	0.99895
4.2	0.99910
4.5	0.99921

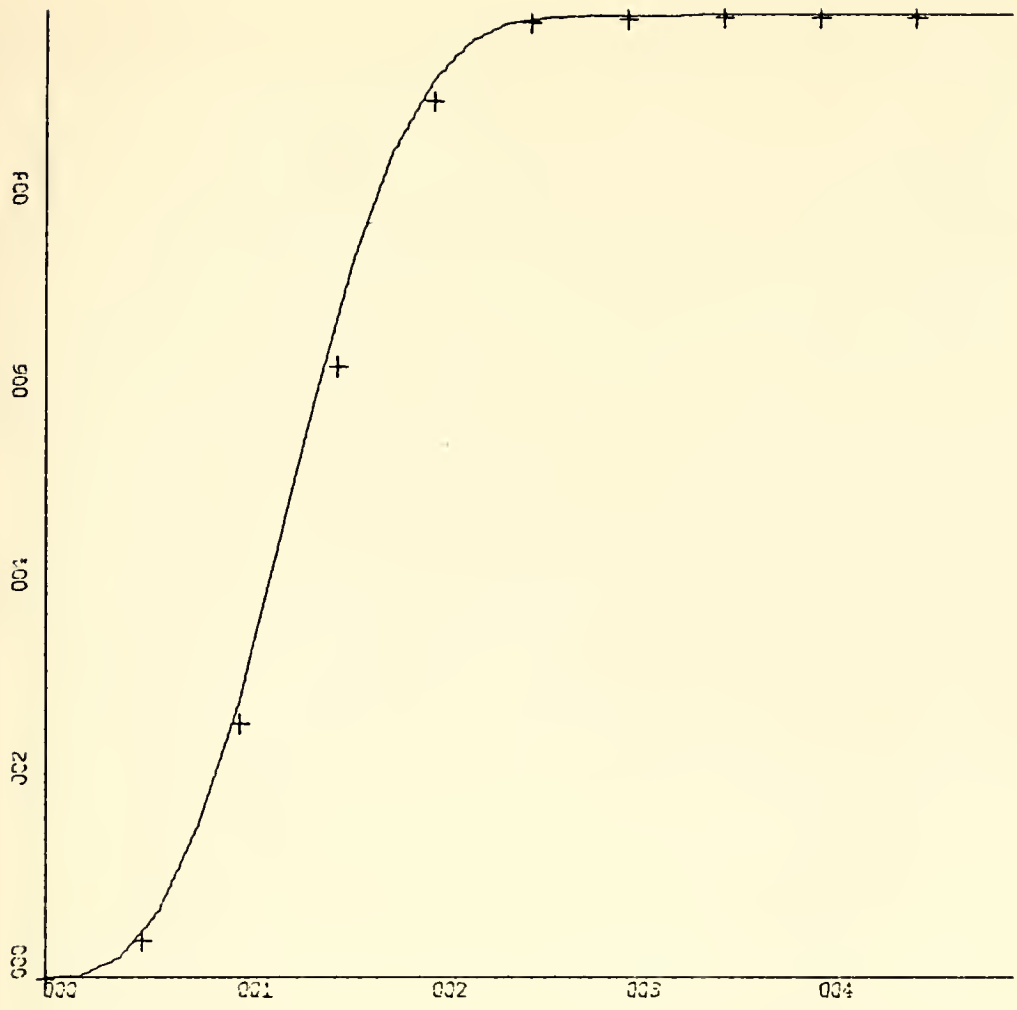
FIGURE 119



FOR DELTA T = 0.4 TFIN = 5.0

T	Q(N)
0.0	-0.00018
0.4	0.02067
0.8	0.15013
1.2	0.41634
1.6	0.71810
2.0	0.91765
2.4	0.98739
2.8	0.99659
3.2	0.99714
3.6	0.99787
4.0	0.99827
4.4	0.99859

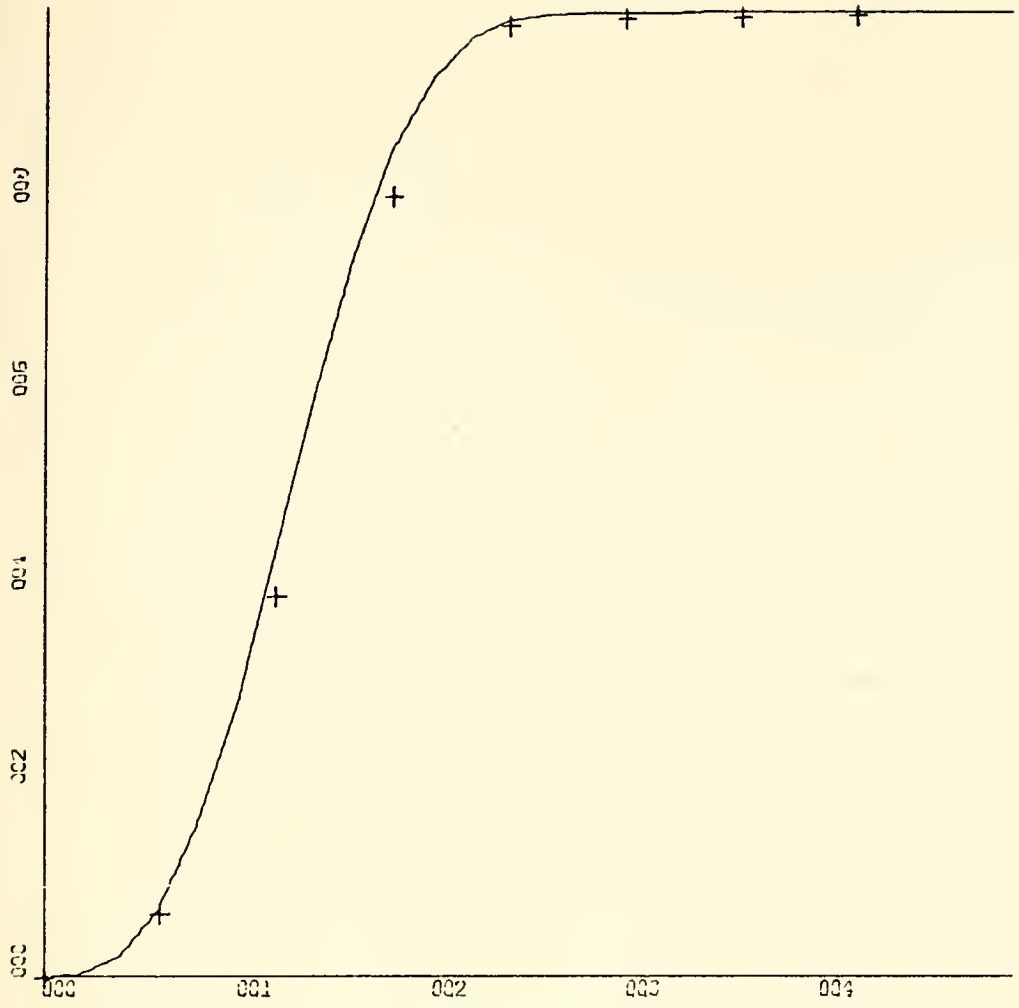
FIGURE 120



FOR DELTA T = 0.5 TFIN = 5.0

T	Q(N)
0.0	-0.00035
0.5	0.03922
1.0	0.26275
1.5	0.63278
2.0	0.90925
2.5	0.99187
3.0	0.99499
3.5	0.99641
4.0	0.99731
4.5	0.99789

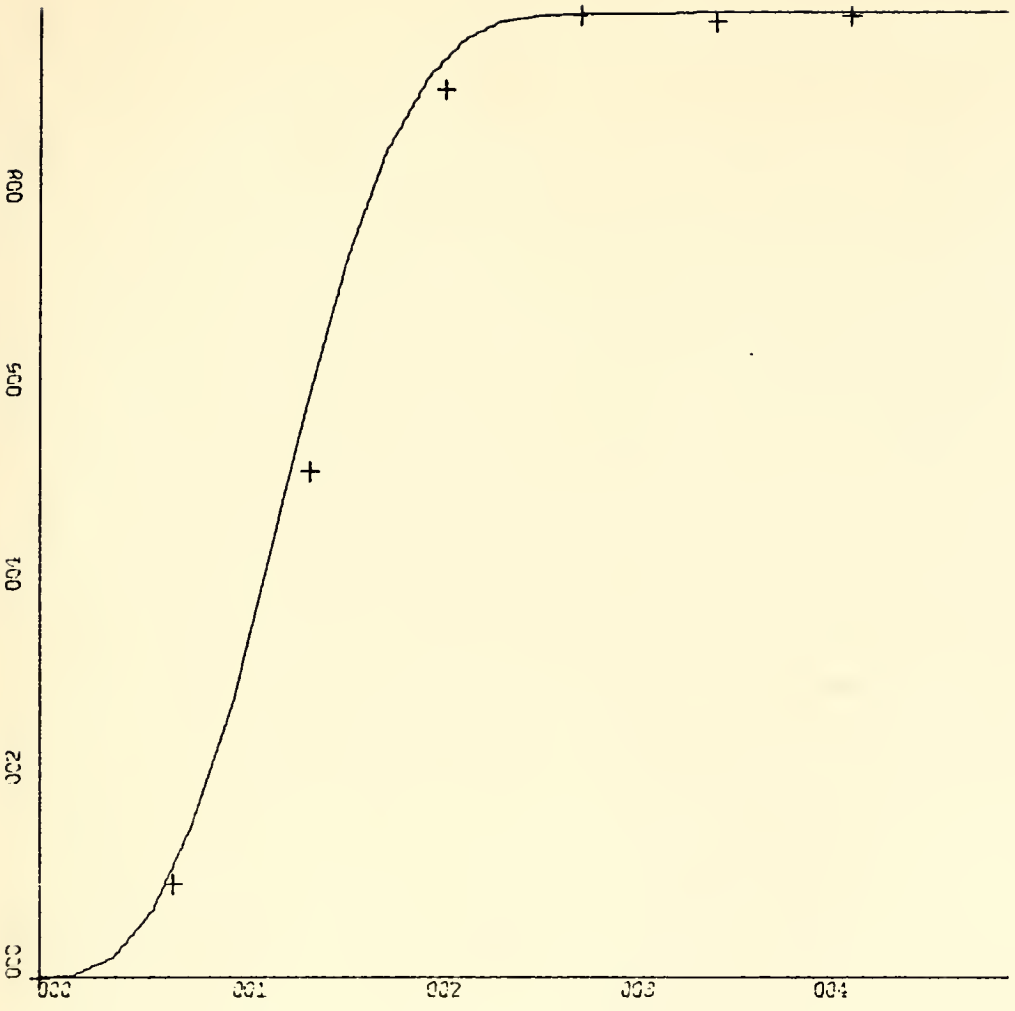
FIGURE 121



FOR DELTA T = 0.6 TFIN = 5.0

T	Q(N)
0.0	-0.00060
0.6	0.06498
1.2	0.39243
1.8	0.80675
2.4	0.98482
3.0	0.99326
3.6	0.99498
4.2	0.99658

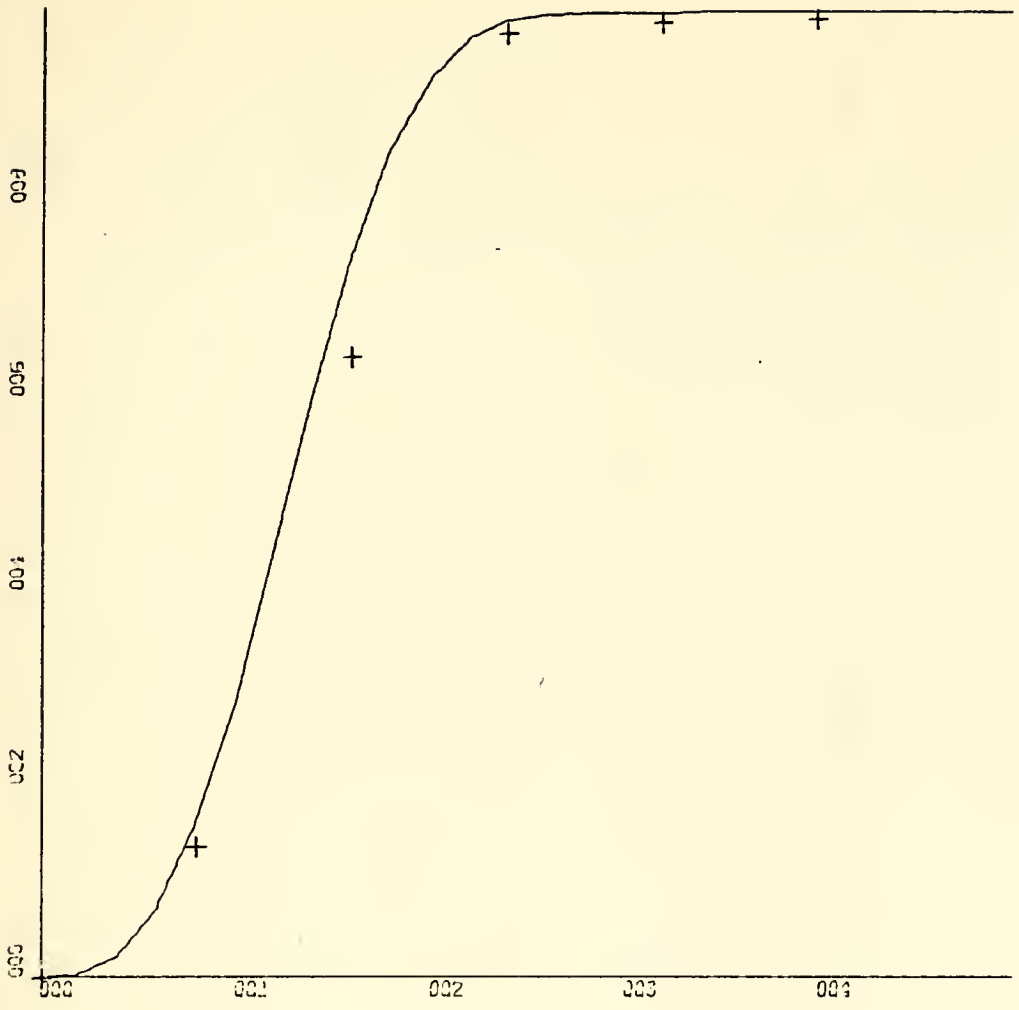
FIGURE 122



FOR DELTA T = 0.7 TFIN = 5.0

T	Q(N)
0.0	-0.00095
0.7	0.09760
1.4	0.52265
2.1	0.91859
2.8	0.99655
3.5	0.99033
4.2	0.99646

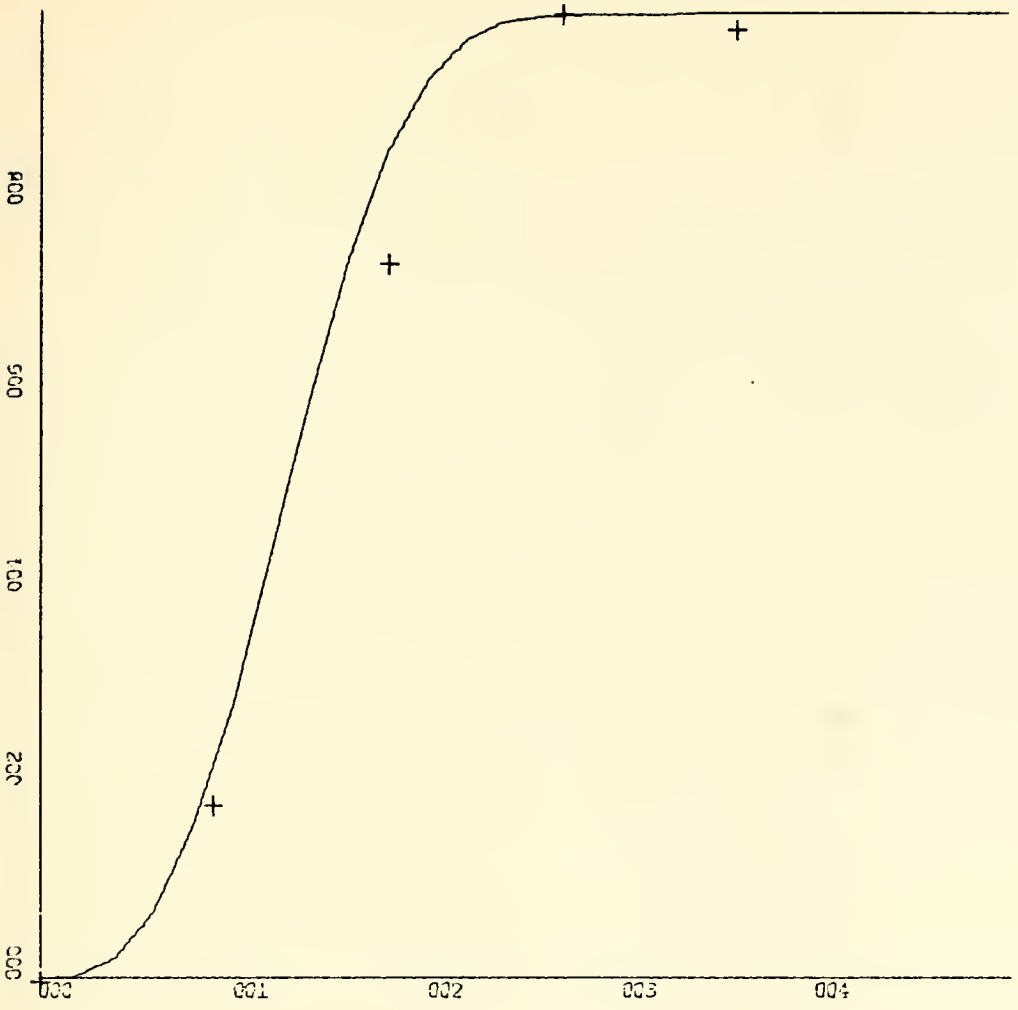
FIGURE 123



FOR DELTA T = 0.8 TFIN = 5.0

T	Q(N)
0.0	-0.00142
0.8	0.13588
1.6	0.64020
2.4	0.97679
3.2	0.98920
4.0	0.99299

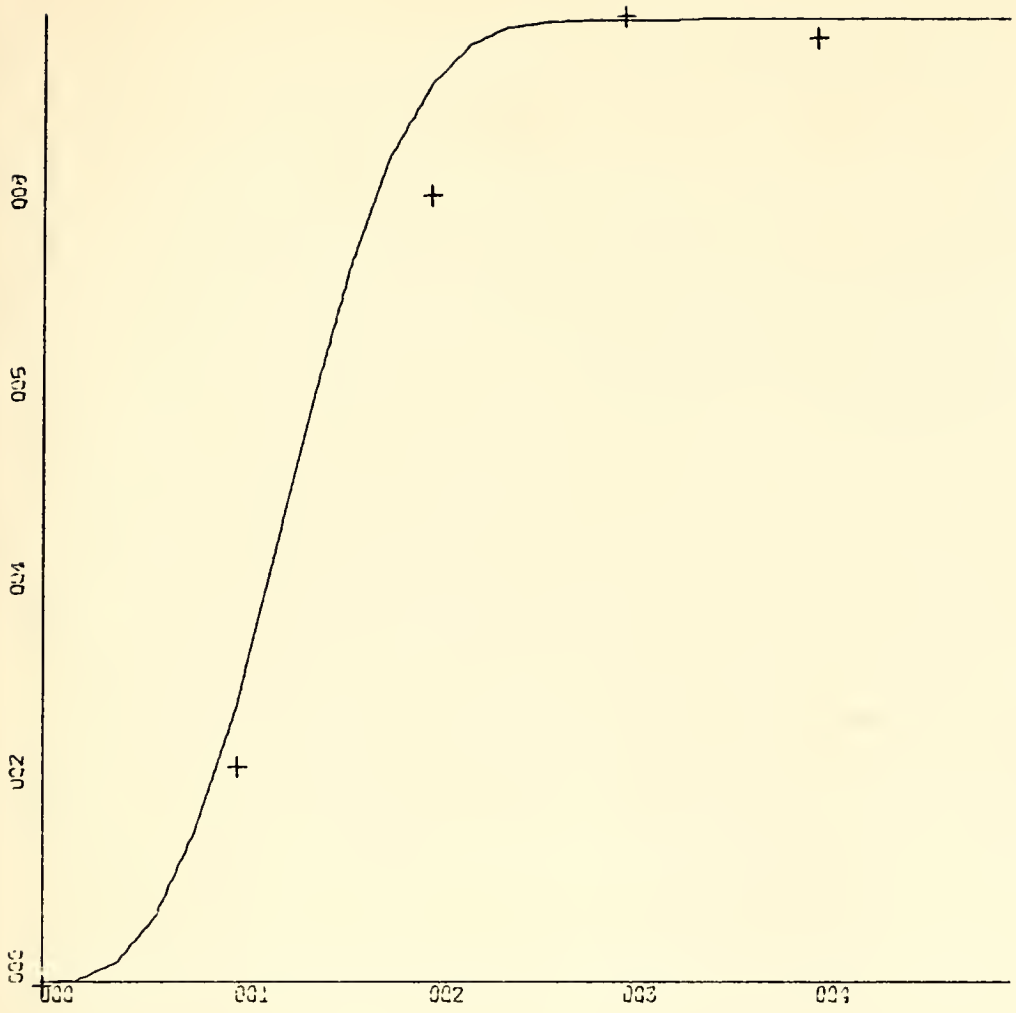
FIGURE 124



FOR DELTA T = 0.9 TFIN = 5.0

T	Q(N)
0.0	-0.00202
0.9	0.17809
1.8	0.73807
2.7	0.99964
3.6	0.98234

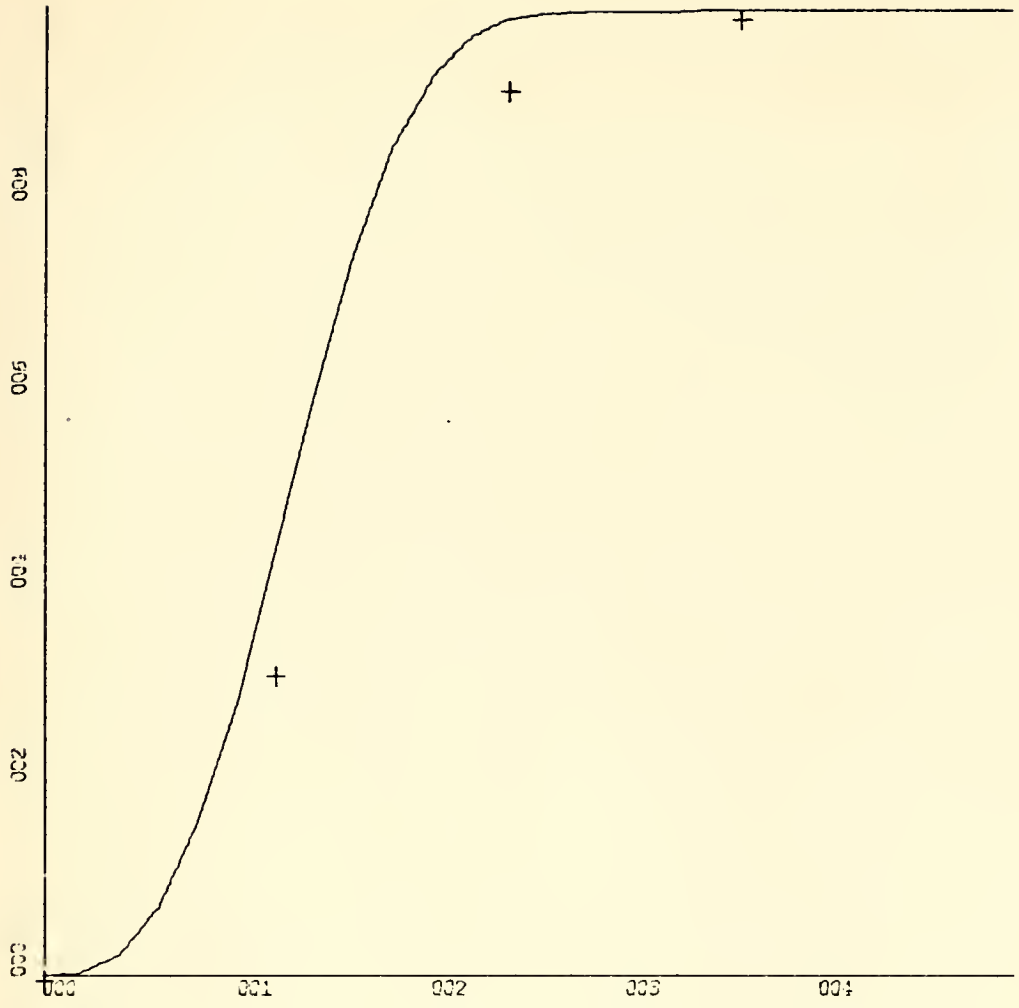
FIGURE 125



FOR DELTA T = 1.0 TFIN = 5.0

T	Q(N)
0.0	-0.00278
1.0	0.22222
2.0	0.81481
3.0	1.00337
4.0	0.98017

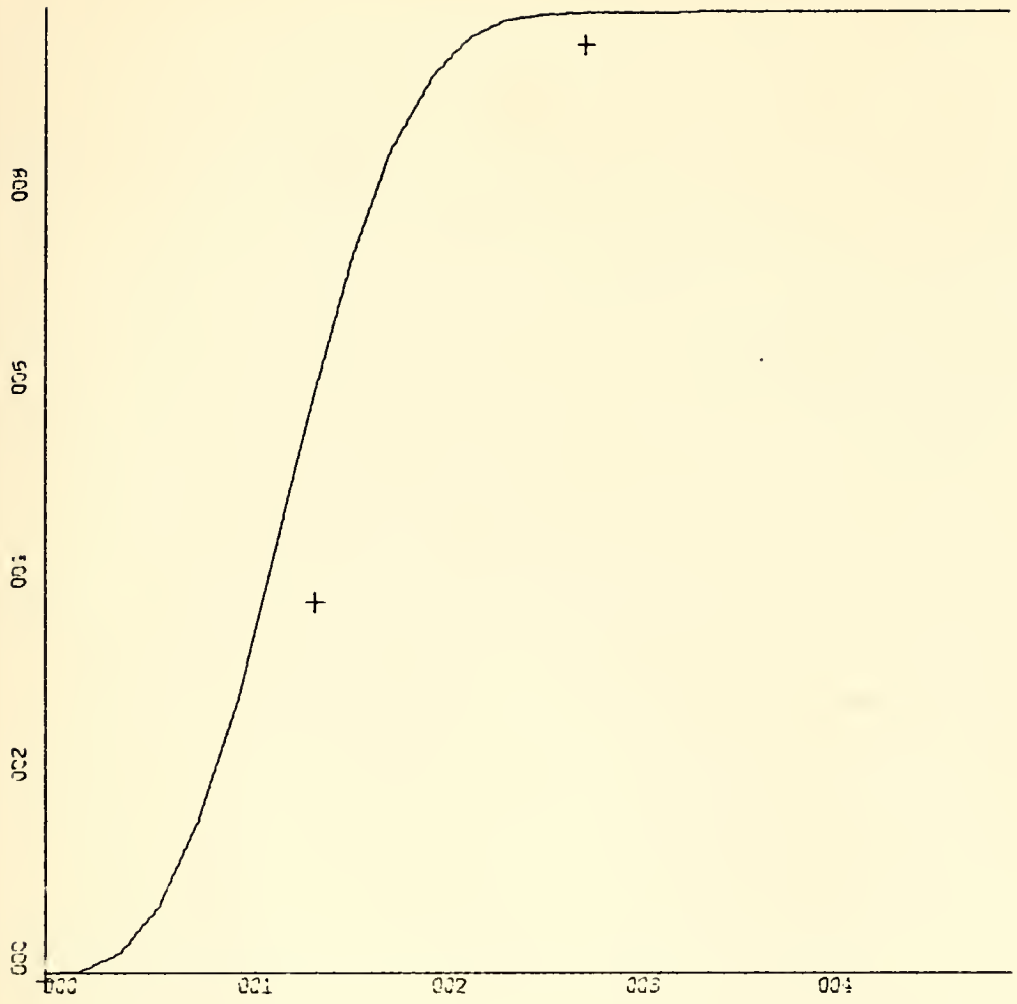
FIGURE 126



FOR DELTA T = 1.2 TFIN = 5.0

T	Q(N)
0.0	-0.00480
1.2	0.30901
2.4	0.91428
3.6	0.99117

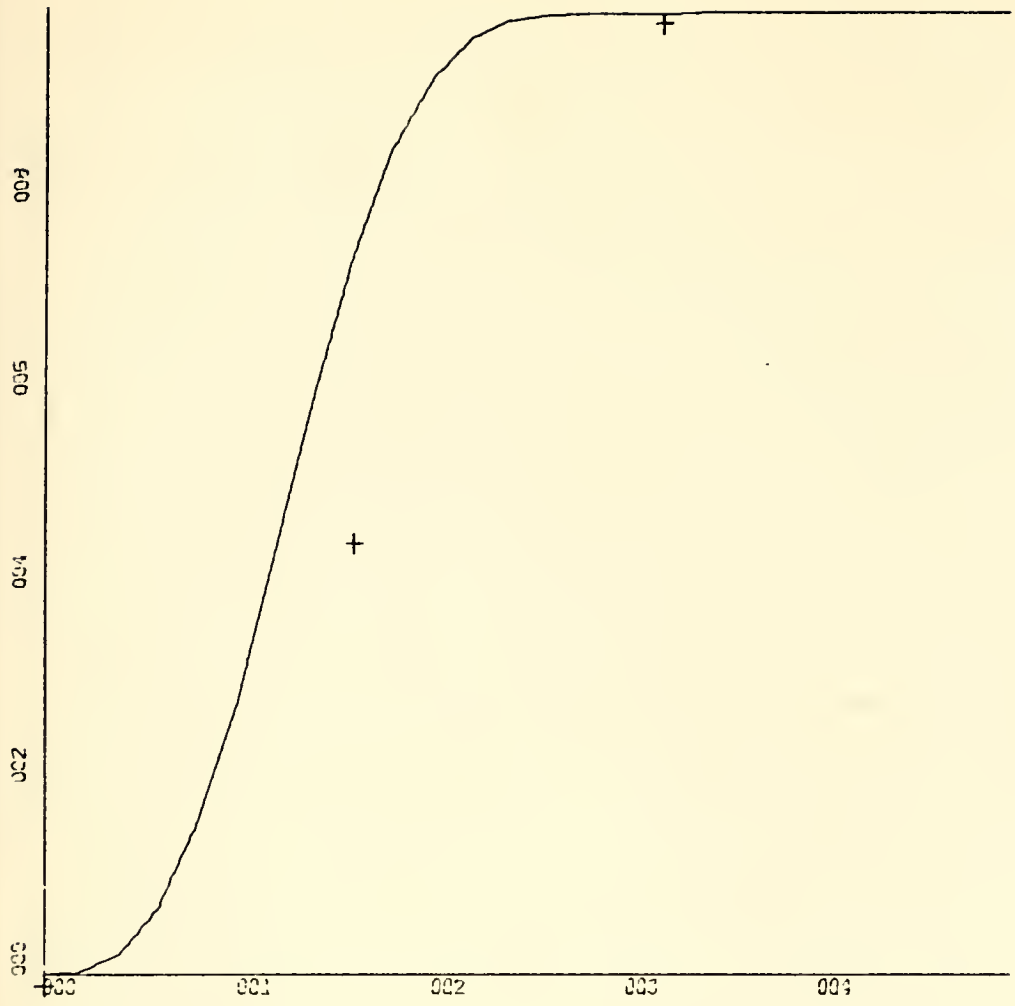
FIGURE 127



FOR DELTA T = 1.4 TFIN = 5.0

T	Q(N)
0.0	-0.00762
1.4	0.38561
2.8	0.96474

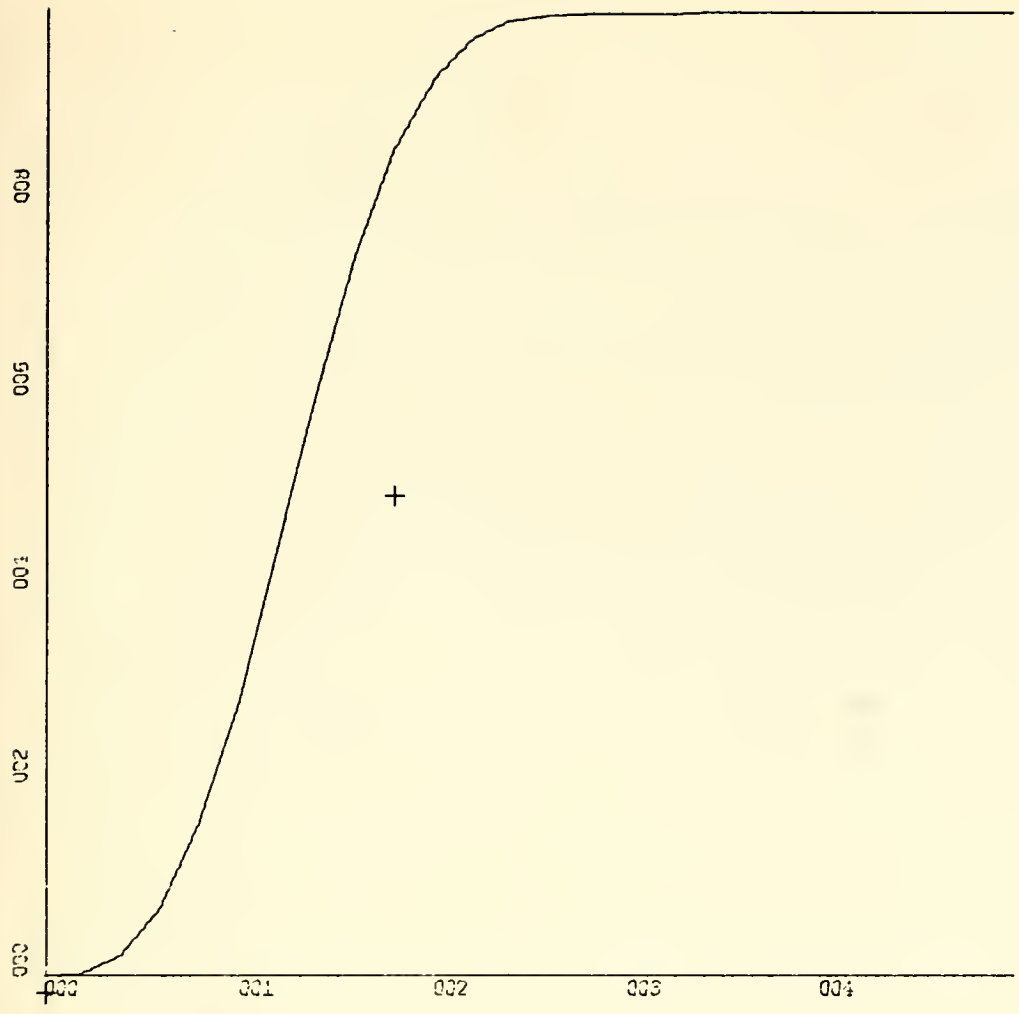
FIGURE 128



FOR DELTA T = 1.6 TFIN = 5.0

T	Q(N)
0.0	-0.01138
1.6	0.44794
3.2	0.98867

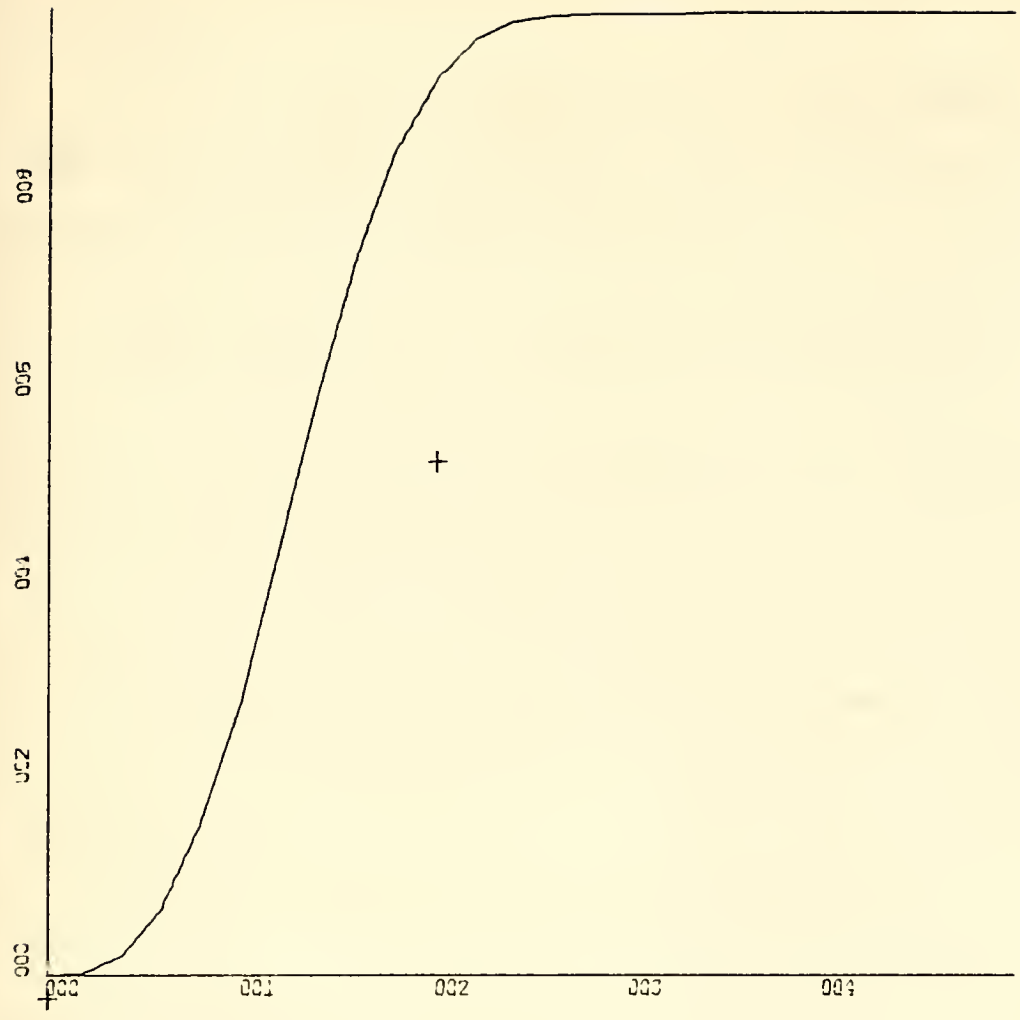
FIGURE 129



FOR DELTA T = 1.8 TFIN = 5.0

T	Q(N)
0.0	-0.01620
1.8	0.49642

FIGURE 130



FOR DELTA T = 2.0 TFIN = 5.0

T	Q(N)
0.0	-0.02222
2.0	0.53333

FIGURE 131

E. CASE 10

No one of the previous cases has a time varying coefficient which is exponential. So, from Ref. 2 this case was extracted. The equation is:

$$C \frac{de}{dt} + \frac{e}{r(t)} = 0 \qquad \text{being } r(t) = R \exp\left(\frac{t}{T_1}\right)$$

where T_1 is a factor in the variation of r. Dividing all the terms by C yields:

$$\frac{de}{dt} + \frac{1}{T_2} \exp\left(\frac{-t}{T_1}\right) e = 0$$

where $T_2 = RC$.

For this example T_1 and T_2 were made equal to one and $e_{(t=0)} = 1.0$. Rewriting the equation and expressing its Laplace transform in powers of s^{-1}:

$$\dot{e} + Ce = 0 \qquad C = e^{-t}$$

$$sE - 1 + CE = 0$$

$$E = \frac{1}{s + C} = \frac{1}{s} - \frac{Cs^{-2}}{1 + Cs^{-1}}$$

If the rules in Example 2, Section IIB were followed, substituting the z-forms of s^{-k}, dividing by T and adding the final results of the quotient to 1.0 which is the time function for 1/s:

$$E_A^* = \frac{-C\,\frac{T^2}{12}\,\frac{1 + 10z^{-1} + z^{-2}}{(1 - z^{-1})^2}}{1 + C\,\frac{T}{2}\,\frac{1 + z^{-1}}{1 - z^{-1}}}$$

For T = 0.1

$$E_A^* = \frac{-0.1 - z^{-1} - 0.1z^{-2}}{(12/C + 0.6) - (24/C)z^{-1} + (12/C - 0.6)z^{-2}}$$

Dividing the denominator into the numerator and adding one to the result one gets

$$0.99206 + 0.91412z^{-1} + 0.85007z^{-2} + \cdots$$

However, looking at the graph (Fig. 132) the precise and the approximated solution start diverging after a while. This is due to the fact that in

$$E = \frac{1}{s + C}$$

if C goes to zero the equation becomes

$$E = 1/s$$

which is the final value that the approximated solution approaches.

This fact is very important because it leads to these important conclusions:

1. If C_i goes to zero with time, first check the Laplace transform and the initial equation substituting C_i by zero. If by inspection they do not lead to the same type of solution, this method does not apply.

2. Verify, before doing the division if any of the varying coefficients, unexpectedly, goes to infinity; by dividing by zero, for instance.

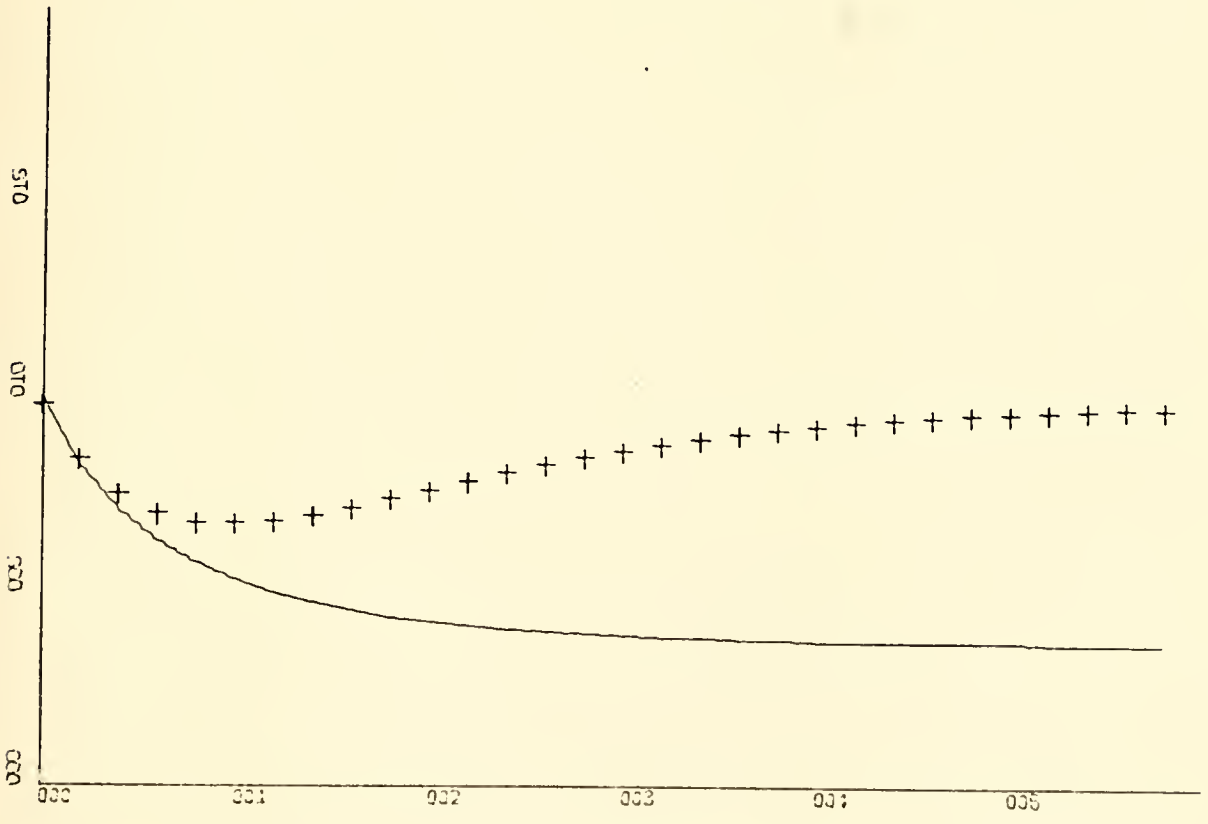


Figure 132

X-scale = 1.0 units/inch
Y-scale = .5 units/inch

— Precise solution
\+ Approximate solution

IV. CONCLUSIONS

This technique, for the successfully area covered in this paper (linear differential equations with time varying coefficients), showed to have a good precision even for relatively large iteration steps. The labor necessary to complete a problem by handwork is not very involved and needs only a small algorithm for computer solution with C.P.U. times much smaller than existing numerical methods.

However, with time varying coefficients that go to zero with time, attention must be paid to the application of this method. As a general rule, this method does not apply.

In some cases, the final equation for the approximate solution, before the division is performed, must be altered in order to not have time varying coefficients (referred as C_i in Section IIC) in the numerator. Caution is necessary to verify, before starting the division steps if any varying coefficient unexpectedly has a value of infinity; by division by zero when substituting a particular value of C_i.

The technique tested in this paper does not apply to the case of linear differential equations with varying coefficients only. Linear differential equations with constant coefficients and nonlinear differential equations are, as well, solved by this technique.

These areas, in the opinion of the author, are valuable as topics for other theses.

LIST OF REFERENCES

1. R. Boxer and S. Thaler, "A Simplified Method of Solving Linear and Nonlinear Systems," Proceedings of the IRE, p. 89-101, January 1956.

2. Cunningham, W.J., Introduction to Nonlinear Analysis, p. 245-247, McGraw-Hill, 1958.

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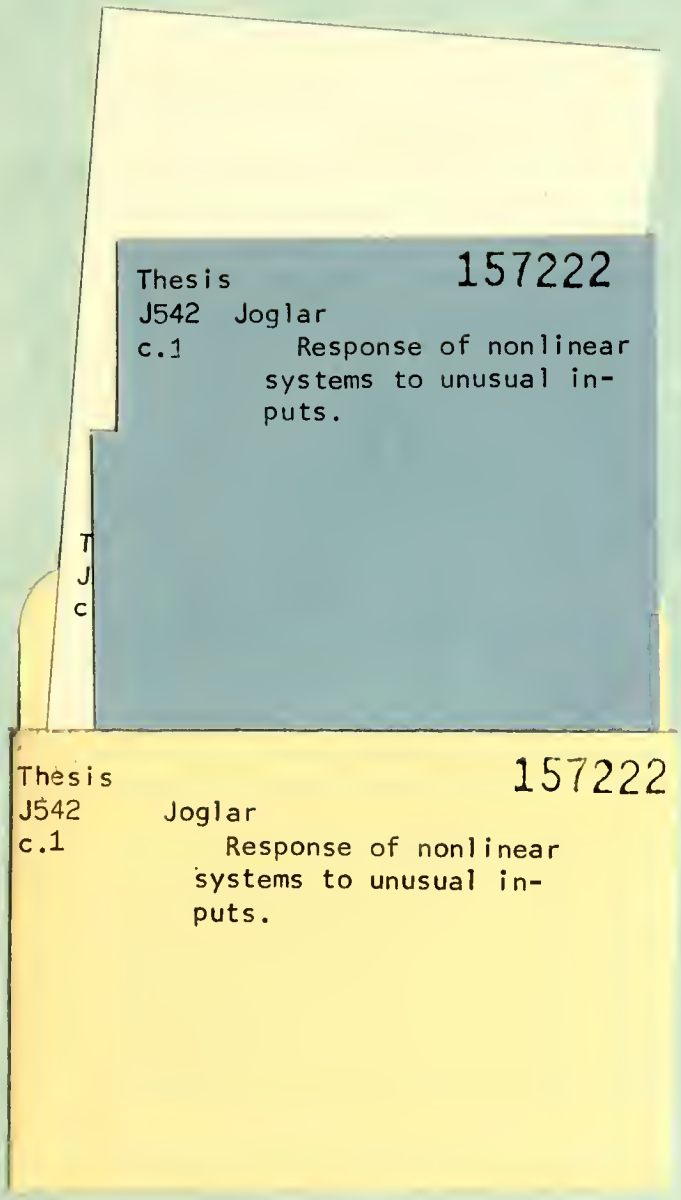

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